

Another Characterization of Parabolic Subgroups

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1. INTRODUCTION

Let G be a reductive algebraic group defined over an algebraically closed field k . Let H be a closed connected subgroup of G containing a maximal torus T of G . In [13] it was shown (at least in characteristic zero) that the parabolic subgroups of G can be characterized among all such subgroups H by a certain finiteness property of the induction functor $(-)|_H^G$ and its derived functors $L_{H,G}^n(-)$. This theme is continued in the present paper, where it is shown that the parabolic subgroups can be characterized by yet another familiar property of the induction functor, at least in characteristic zero. We also obtain several results which are independent of the characteristic by added hypotheses on H , or by using the restriction functor instead of induction.

After describing the relevant property below, Section 2 begins by studying induction from a special type of subgroup H . Namely, H is the semidirect product of T with U_J , where U_J is the unipotent radical of a standard parabolic subgroup P_J . This type of subgroup was also found to be useful in [13].

Section 3 contains the main result, involving the restriction functor. One first reduces the question to the case when $L_I \subseteq H \subseteq P_I$ for some parabolic P_I with Levi factor L_I , then proves the result with this added hypothesis on H .

In Section 4 we work with the induction functor $(-)|_H^G$. If $L_I \subseteq H \subseteq P_I$ holds, we obtain the desired result, but we do not have at this time a way to reduce the question to this case. However, if the characteristic is 0 we can reduce the question to the case of a solvable subgroup H , where the above condition is automatic (taking I to be empty).

In the appendix we apply these results and those of [13] to a special case which was first studied in [6, 14]. The appendix also contains a short

proof of the Mackey decomposition theorem for two parabolics [6, Example 4.5] which is independent of the rest of the paper.

Let V be a rational G -module and recall that the socle $\text{Soc}(V)$ of V is the sum of the irreducible submodules of V , and is the unique maximal completely reducible submodule of V . Suppose that $V=0$ or that $\text{Soc}(V)$ is irreducible. Then we will say that V is a *topheavy* module (because any remaining composition factors must live “on top” of this irreducible submodule). Let F be a functor taking rational G -modules to rational modules for another group H . We will say that F is a *topheavy* functor if $F(V)$ is a topheavy H -module whenever V is an irreducible G -module.

Note that an irreducible module is necessarily topheavy while a topheavy module is necessarily indecomposable. (In fact, V is topheavy iff every submodule is indecomposable.) Of course, sometimes these concepts coincide, except that 0 is not usually considered to be irreducible. For example, if $\text{char } k=0$ and H is reductive, then Weyl’s complete reducibility theorem implies that any indecomposable module is actually irreducible. In this case, F is a topheavy functor iff it “preserves irreducibility.”

As an example of a topheavy functor, let G be a reductive group and let $H=L_J$, the Levi factor of the standard parabolic P_J . Let U_J be the unipotent radical of P_J . Let $F:\text{RAT}(G)\rightarrow\text{RAT}(L_J)$ be the functor $V\rightarrow V^{U_J}$ which assigns to V the fixed point space for the action of U_J on V . Then the main result of [12] shows that F is topheavy; in fact F satisfies the stronger condition that it preserve irreducibility in all characteristics.

More interesting examples of topheavy functors are the induction functors $(-)|_H^G$ when H is a parabolic subgroup of G (see [14, Proposition 5.1]). We mention the fixed points functor $(-)^{U_J}$ because its topheaviness influences the possible topheaviness of $(-)|_H^G$. Indeed [12] is a major ingredient in the results of Section 2 and also plays a role in Section 3.

The classic application of the topheaviness of an induction functor is the case when $H=B$, a Borel subgroup of G . When $\text{char } k=0$, this is the standard construction of the G -irreducibles. Let λ be a character of T , regarded as a one-dimensional (irreducible) B -module. Then $\lambda|_B^G$ is topheavy and hence G -irreducible. As λ ranges over all the characters of T , we obtain all the irreducibles for G . Moreover, $\lambda|_B^G=0$ unless λ is negative dominant. The fact that $(-)|_B^G$ is topheavy and kills λ unless $-\lambda$ is dominant is merely an application of the uniqueness of B -stable lines in a G -irreducible, via Frobenius reciprocity. Note that such a B -stable line is actually the socle of the G -irreducible in question, regarded as a B -module. We conclude this section with a generalization of this observation.

Let H be a closed connected subgroup of G containing T . As described in [13], H has a Levi decomposition $H=L\cdot U_1$, and the irreducible H -modules are the same as the irreducible L -modules. Since L is reductive, these irreducibles can be listed according to the standard highest weight

theory. In particular we will use $S_L(\lambda)$ to denote the irreducible L -module with highest weight λ , where λ is a character of T which is dominant for L 's root system. Its dual is denoted $M_L(-\lambda)$, which has $-\lambda$ as its lowest weight. Let A_+ denote the dominant characters of T , and let A_+^L denote the weights of A which are dominant for L 's root system. We will take $B \cap L$ as our Borel subgroup of L , where B is a Borel subgroup of G . This ensures that $A_+ \subseteq A_+^L$.

PROPOSITION 1. *Let G be a reductive group, and H as above. Assume that for every dominant character λ of T , the G -module $M(-\lambda)$ appears in the socle of $M_L(-\lambda)|_H^G$. Then restriction $(-)|_H$ is a topheavy functor if and only if induction $(-)|_H^G$ is a topheavy functor which kills every H -irreducible whose lowest weight is not negative dominant.*

Proof. If λ is dominant, we are assuming that $\text{Hom}_G(M(-\lambda), M_L(-\lambda)|_H^G)$ is nonzero. If induction kills any $M_L(-\mu)$ when μ is not a dominant weight then $\text{Hom}_G(M(-\lambda), M_L(-\mu)|_H^G) = 0$ for all such μ . If induction is topheavy then $\text{Hom}_G(M(-\lambda), M_L(-\mu)|_H^G) = 0$ if μ is dominant but not equal to λ and moreover $\text{Hom}_G(M(-\lambda), M_L(-\lambda)|_H^G)$ is one-dimensional. Thus we have shown that $\text{Dim Hom}_G(M(-\lambda), M_L(-\mu)|_H^G) = \delta_{\lambda\mu}$ (Kronecker delta) for all $\lambda \in A_+$ and all $\mu \in A_+^L$. But then Frobenius reciprocity implies that

$$\begin{aligned} \text{Hom}_G(M(-\lambda), M_L(-\mu)|_H^G) &\cong \text{Hom}_H(M(-\lambda), M_L(-\mu)) \\ &\cong \text{Hom}_H(S_L(\mu), S(\lambda)), \end{aligned}$$

so for each $S(\lambda)$ the socle of $S(\lambda)|_H$ is just $S_L(\lambda)$.

Conversely if restriction is topheavy, then as $\text{Hom}_H(S_L(\lambda), S(\lambda)|_H)$ is nonzero, we obtain $\text{Dim Hom}_H(S_L(\mu), S(\lambda)) = \delta_{\lambda\mu}$ for all $\lambda \in A_+$ and all $\mu \in A_+^L$. Thus we may reverse the above argument to obtain that $M_L(-\lambda)|_H^G$ has socle $M(-\lambda)$ if λ is dominant, while $M(-\mu)|_H^G = 0$ if μ is not dominant. ■

As for the higher derived functors of induction, if $\text{char } k = 0$, then $L_{B,G}^n(-)$ is topheavy for all n by the Borel–Weil–Bott theorem [3]. In prime characteristics, Andersen [1] has given examples to show that $L_{B,G}^2(-)$ need not be topheavy, although $L_{B,G}^1(-)$ always is topheavy. However, Humphreys [9] suggests that $L_{B,G}^n(-)$ should be “generically” topheavy for all n and hence “generically” bottomheavy for all n also, by Serre duality. That is, for generic choices of λ , $L_{B,G}^n(\lambda)$ should have both an irreducible socle and an irreducible head. (The head of a module is the dual notion to the socle, namely the unique maximal completely reducible quotient.)

When H is not a Borel subgroup, even less is known. For our purposes, we may avoid the higher derived functors. We will show in many cases that the topheaviness of $(-)|_H^G$ alone is enough to force H to be parabolic. The notation throughout agrees with [14, 13].

2. A SPECIAL CASE

Let P_J be the standard parabolic subgroup of G determined by a subset J of the simple roots Δ . Let U_J be the unipotent radical of P_J . Since T normalizes U_J , we may form the semidirect product $T \cdot U_J$, which is isomorphic via multiplication to a closed connected subgroup H of G . The Levi decomposition of H is just $T \cdot U_J$, so H is a subgroup of $T \cdot U = B$. In particular, H is a solvable group so every irreducible rational H -module is one-dimensional, on which T acts via some character λ .

In this section, we will examine the induction functor $(-)|_H^{P_K}$ whenever $J \subseteq K \subseteq \Delta$. If J is empty then $U_J = U$ and $H = B$. This case is well understood, so we may ignore it. On the other extreme, if $J = \Delta$ then U_J is trivial and $H = T$. Using reciprocity it is quite easy to check that induction $(-)|_T^G$ is not topheavy so we also ignore this case and assume that J is a proper subset of Δ .

Recall (see [14]) that $M_J(\mu)$ denotes the irreducible P_J -module with lowest weight μ , where μ is the negative of a J -dominant character of T . If $I \subseteq J$ then let $U_{J,I}$ denote the unipotent radical of $P_I \cap L_J$, where L_J is the Levi factor of P_J . It contains the one-dimensional root groups U_γ for $\gamma \in \Phi_J^+ - \Phi_I^+$.

LEMMA 2.1. *Let $I \subseteq J \subseteq K$ be a chain of subsets of Δ , and suppose that λ is a J -dominant character of T . Then we have isomorphisms of P_I -modules:*

$$(M_J(-\lambda)^*)^{U_{K,I}} \cong (M_J(-\lambda)^{U_{\bar{K},I}})^* \cong M_I(-\lambda)^*.$$

Proof. In [12] it is shown that $S(\lambda)^{U_I} \cong S_I(\lambda)$, and similar arguments show that $M(-\lambda)^{U_I^-} \cong M_I(-\lambda)$, where U_I^- has root groups with roots from $\Phi^- - \Phi_I^-$. There is a short proof of this result based on induced modules in [11], and in [14] this is extended to obtain that $M_J(-\lambda)^{U_{J,I}} \cong M_I(-\lambda)$ if λ is J -dominant. Similarly, we obtain that $S_J(\lambda)^{U_{J,I}} \cong S_I(\lambda)$.

Now if $I \subseteq J \subseteq K$, then $U_{K,J}$ is normal in $U_{K,I}$ and the quotient is isomorphic to $U_{J,I}$. Moreover $U_{K,J} \subseteq U_J$, which acts trivially on $S_J(\lambda)$ and similarly $U_{\bar{K},J}$ acts trivially on $M_J(-\lambda)$. Hence $(M_J(-\lambda)^*)^{U_{K,I}} \cong (S_J(\lambda)^{U_{K,J}})^{U_{J,I}} \cong S_J(\lambda)^{U_{J,I}} \cong S_I(\lambda) \cong M_I(-\lambda)^* \cong (M_J(-\lambda)^{U_{\bar{J},I}})^* \cong ((M_J(-\lambda)^{U_{\bar{K},J}})^{U_{\bar{J},I}})^* \cong (M_J(-\lambda)^{U_{\bar{K},I}})^*$, as claimed. ■

Now let Q_J be the root lattice for the root system Φ_J , and let Q_J^+ be the subset of all nonnegative integral combinations of roots in J . Also let A_+^K denote the K -dominant characters of T . If M is a module and n is a nonnegative integer, let M^n denote a direct sum of n copies of M .

THEOREM 2.2. *Let G be a reductive group and let H be the subgroup $T \cdot U_J$, for some $J \subseteq \Delta$. Let K be any subset of Δ containing J . Then:*

$$(a) \quad \text{Soc}_{P_K}(-\lambda|_H^{P_K}) \cong \bigoplus_{\mu \in (\lambda + Q_J^+) \cap A_+^K} M_K(-\mu)^{m(\mu, \lambda)} \\ \cong \text{Soc}_{L_K}(-\lambda|_H^{P_K}|_{L_K}); \text{ where } m(\mu, \lambda) = \dim S_J(\mu)_\lambda.$$

(b) *Let $d_{H, P_K}(\lambda) = \sum_{\mu \in (\lambda + Q_J^+) \cap A_+^K} m(\mu, \lambda)$, so $d_{H, P_K}(\lambda)$ is the number of summands in this socle. If J is empty then (a) reduces to [14, Proposition 4.3], $d_{B, P_K}(\lambda) = 1$ for all $\lambda \in A_+^K$. If J is nonempty then:*

(i) *If $\text{char } k = 0$ then for any dominant λ which is sufficiently far from the walls of the dominant chamber we have $d_{H, G}(\lambda) \geq 2$. In particular induction $(-)|_H^{P_K}$ is not a topheavy functor for any K between J and Δ .*

(ii) *If $\text{char } k = p$, suppose λ is a dominant character with the property that there exists an $\alpha \in J$ with $\langle \lambda, \alpha \rangle = np - 1$ for some $n > 0$ and such that $\lambda + \alpha$ is also dominant. Then $d_{H, G}(\lambda) \geq 2$. In particular $(-)|_H^{P_K}$ is not a topheavy functor for any K between J and Δ .*

Proof. For (a), observe that by definition of the socle we have $\text{Soc}(-\lambda|_H^{P_K}) \cong \bigoplus_{\mu \in A_+^K} M_K(-\mu)^{n(\mu, \lambda)}$, where $n(\mu, \lambda) = \dim \text{Hom}_{P_K}(M_K(-\mu), -\lambda|_H^{P_K})$. But by reciprocity and the fact that U_J acts trivially on $-\lambda$ we obtain:

$$\text{Hom}_{P_K}(M_K(-\mu), -\lambda|_H^{P_K}) \cong \text{Hom}_H(M_K(-\mu), -\lambda) \\ \cong (\text{Hom}(M_K(-\mu), -\lambda)^{U_J})^T \\ \cong (\text{Hom}(M_K(-\mu), k)^{U_J} \otimes -\lambda)^T \\ \cong ((M_K(-\mu)^*)^{U_J} \otimes -\lambda)_0 \cong (M_K(-\mu)^*)_\lambda^{U_J} \otimes -\lambda.$$

But Lemma 2.1 applied to $J \subseteq K \subseteq \Delta$ implies that $(M_K(-\mu)^*)^{U_J} \cong M_J(-\mu)^*$, so we obtain that $n(\mu, \lambda) = \dim \text{Hom}_{P_K}(M_K(-\mu), -\lambda|_H^{P_K}) = \dim M_J(-\mu)_\lambda^* = \dim S_J(\mu)_\lambda = m(\mu, \lambda)$. Moreover, $S_J(\mu)_\lambda = 0$ unless $\lambda \leq \mu$ in the partial order determined by J , so $m(\mu, \lambda) = 0$ unless $\mu \in \lambda + Q_J^+$.

Next, every homomorphism of P_K -modules is also L_K -equivariant, so $\text{Hom}_{P_K}(M_K(-\mu), -\lambda|_H^{P_K}) \subseteq \text{Hom}_{L_K}(M_K(-\mu), -\lambda|_H^{P_K}|_{L_K})$. As $U_K \subseteq U_J \subseteq H$, $HL_K = P_K$ and [6, 4.1] applies to give that $-\lambda|_H^{P_K}|_{L_K} \cong -\lambda|_{H \cap L_K}|_{L_K}^{L_K}$, so by reciprocity we have $\text{Hom}_{L_K}(M_K(-\mu), -\lambda|_H^{P_K}|_{L_K}) \cong \text{Hom}_{H \cap L_K}(M_K(-\mu), -\lambda)$. Moreover, $H \cap L_K \cong T \cdot U_{K, J}$, so this further simplifies to

$$(\text{Hom}(M_K(-\mu), k)^{U_{K, J}} \otimes -\lambda)_0 \cong (M_K(-\mu)^*)_\lambda^{U_{K, J}} \otimes -\lambda.$$

But this is just $S_J(\mu)_\lambda \otimes -\lambda$ by Lemma 2.1 applied to $J \subseteq K \subseteq K$. It follows that $\text{Soc}_{L_K}(-\lambda|_H^{P_K}|_{L_K})$ has the same expression as the above one for $\text{Soc}_{P_K}(-\lambda|_H^{P_K})$, completing the proof of (a).

Now suppose $\text{char } k = 0$ and J is nonempty. Choose $\lambda \in A_+$ such that $\langle \lambda, \beta \rangle \geq 3$ for all $\beta \in \Delta$. Then $\langle \lambda + \alpha, \beta \rangle \geq 0$ for all simple roots α and β , since $\langle \alpha, \beta \rangle \geq -3$ for any pair of simple roots in any root system. Thus $\lambda + \alpha$ is also dominant for any simple root α . Select any α from J . Let $\mu = \lambda + \alpha$ and observe that $\mu \in A_+ \cap (\lambda + Q_J^+)$, hence $M(-\mu)$ is a nonzero irreducible G -module appearing in the socle of $-\lambda|_H^G$ by part (a). Indeed $\dim S_J(\mu)_\lambda$ is nonzero because in characteristic zero, the set of weights for an irreducible P_J -module is a saturated set (see [8]) of weights (for the root system Φ_J) with highest weight μ , and λ belongs to this saturated set. Of course, part (a) implies that $M(-\lambda)$ is also part of the socle, so $d_{H,G}(\lambda) \geq 2$. But $A_+ \subseteq A_+^K \subseteq A_+^J$, so the definition of $d_{H,G}$ reveals that $d_{H,G}(\lambda) \leq d_{H,P_K}(\lambda) \leq d_{H,P_J}(\lambda)$. Thus $d_{H,P_K}(\lambda) \geq 2$ for all K containing J , showing that $(-)|_H^{P_K}$ is never topheavy.

Now assume that $\text{char } k = p$, and locate a dominant character λ which meets the hypotheses of the theorem. For example, let $\mu_r = (p^r - 1)\rho$, the r th Steinberg character. It has the property that $\langle \mu_r, \alpha \rangle = p^r - 1$ for all $\alpha \in \Delta$, and both μ_r and $\mu_r + \alpha$ are dominant for large r . Then we claim that $-\lambda$ is not strongly linked (see [2]) to $-(\lambda + \alpha)$ via $W_{\{\alpha\},p}$, the affine Weyl group of type A_1 for the minimal parabolic P_α . Admitting this for the moment, observe that all the composition factors of $-(\lambda + \alpha)|_B^{P_\alpha}$ have the form $M_{\{\alpha\}}(\mu)$ with μ strongly linked (via $W_{\{\alpha\},p}$) to $-(\lambda + \alpha)$, hence $\mu \neq -\lambda$. But $-\lambda$ is a weight of $-(\lambda + \alpha)|_B^{P_\alpha}$, so must also be a weight of some composition factor $M_{\{\alpha\}}(\mu)$ with $-(\lambda + \alpha) \leq \mu \leq -\lambda$ in the partial order determined by $\{\alpha\}$. Since we ruled out $\mu = -\lambda$, the only other possibility is $\mu = -(\lambda + \alpha)$, hence $M_{\{\alpha\}}(-(\lambda + \alpha))_{-\lambda} \neq 0$. Since $\alpha \in J$ we may use Smith's theorem [12] to obtain

$$0 \neq M_{\{\alpha\}}(-(\lambda + \alpha))_{-\lambda} \cong M_J(-(\lambda + \alpha))_{-\lambda}^{U_{-\lambda}^J} \subseteq M_J(-(\lambda + \alpha))_{-\lambda}.$$

Thus by part (a) we see that both $M_K(-\lambda)$ and $M_K(-(\lambda + \alpha))$ appear in the socle of $-\lambda|_H^{P_K}$ and so $d_{H,P_K}(\lambda) \geq 2$. This will show that $(-)|_H^{P_K}$ is not topheavy in prime characteristics either.

It remains to show that under these hypotheses $-\lambda$ is not strongly linked to $-(\lambda + \alpha)$ via $W_{\{\alpha\},p}$. Since $-\lambda$ and $-(\lambda + \alpha)$ differ by α , either they lie in the same alcove for the dot action for this affine Weyl group, or they lie in adjacent alcoves. In the former case they are not p -linked and we are done. In the latter case they must differ by a single reflection if they are to be p -linked, so we actually have $-\lambda \uparrow -(\lambda + \alpha)$. That is, there exists an $m \geq 0$ with $\langle -(\lambda + \alpha) - \rho, \alpha \rangle < -mp$ and $-\lambda = s_{\alpha, -m} \cdot (-(\lambda + \alpha))$ (see [2] for this notation). But then using $\langle \lambda, \alpha \rangle = np - 1$ we obtain

$$\begin{aligned}
 -\lambda &= s_\alpha(-(\lambda + \alpha) - \rho) + \rho - mp\alpha \\
 &= -\lambda - \alpha - \langle -(\lambda + \alpha) - \rho, \alpha \rangle \alpha - mp\alpha \\
 &= -\lambda + (1 + np - mp)\alpha.
 \end{aligned}$$

But this says that $0 = 1 + (n - m)p$, which is impossible. ■

COROLLARY 2.3. *Suppose $\text{char } k = 0$ and λ is a character of T . Let $H = T \cdot U_J$ and suppose $J \subseteq K \subseteq \Delta$. Then $-\lambda|_H^{P_K}$ is a completely reducible module, and its formal character is given by*

$$\chi(-\lambda|_H^{P_K}) = \sum_{\mu \in (\lambda + Q_J^+) \cap A_+^K} m(\mu, \lambda) \chi(M_K(-\mu)),$$

where the characters of the irreducibles on the right are given by Weyl’s formula for L_K and the coefficients $m(\mu, \lambda) = \dim S_J(\mu)_\lambda$ may be computed via Kostants multiplicity formula for L_J . (See [8] for these formulas.)

Proof. From Theorem 2.2(a) we have $\text{Soc}_{P_K}(-\lambda|_H^{P_K}) = \text{Soc}_{L_K}(-\lambda|_H^{P_K}|_{L_K})$, but in characteristic zero L_K acts completely reducibly by Weyl’s theorem, so the latter socle must be the entire module. The rest is clear. ■

Observe that the above sum turns out to be finite, by [13, Cor. 2.3].

3. TOPHEAVY RESTRICTION FUNCTORS

Assume G is a reductive algebraic group, T a maximal torus of G , and H a closed connected subgroup of G containing T which we hope to show is parabolic. Recall that H has a Levi decomposition $H = L \cdot U_1$. Consider the case $U_1 = \{1\}$, so $H = L$ is reductive also. We recall some basic facts about L ’s root system. Let $\Phi_L = \{\gamma \in \Phi \mid U_\gamma \subseteq L\}$, where U_γ is the root subgroup associated to γ . Then $\Phi_L = -\Phi_L$ because L is generated by its subgroups of type A_1 , and Φ_L also has the property that if α and β are roots in Φ_L with $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi_L$. Indeed L contains the subgroup generated by U_α and U_β , which contains $U_{\alpha+\beta}$ if $\alpha + \beta$ is a root in Φ . Similarly for $\alpha - \beta$ since $\Phi_L = -\Phi_L$. It follows that Φ_L is a root system (see [8, p. 46, Exercise 7]). Moreover it also follows by induction on the number of summands that if $\alpha_i \in \Phi_L$ and $\sum k_i \alpha_i \in \Phi$ then $\sum k_i \alpha_i \in \Phi_L$, for any integers k_i . In other words $\Phi \cap Z\Phi_L = \Phi_L$, where $Z\Phi_L$ denotes the root lattice for Φ_L .

Now assume that restriction $(-)|_L: \text{RAT}(G) \rightarrow \text{RAT}(L)$ is a tophheavy functor. If this property characterizes parabolics, then G/L would be a projective variety. But since L is reductive, G/L is affine [5]. Hence G/L is a single point (provided G is connected) and $L = G$. Thus for reductive sub-

groups, we expect that restriction is never topheavy for proper subgroups. We prove this first:

LEMMA 3.1. *Let L be a reductive subgroup of G . Then restriction $(-)|_L$ is a topheavy functor iff $L = G$.*

Proof. If $L = G$, we obtain the identity functor, which is topheavy. Conversely if $L \neq G$, we will produce an irreducible G -module whose restriction to L is decomposable, whence restriction is not a topheavy functor.

First, let V be any rational G -module. Let μ and λ be weights of V and define $\lambda \sim \mu$ to mean that $\lambda - \mu \in Z\Phi_L$. One easily checks that this defines an equivalence relation on the set $\Lambda(V)$ of weights of V . Observe that each equivalence class \mathcal{C} is stable under the action of L 's Weyl group W_L . (Induct on the length of a word in W_L .) Now let $\{\mathcal{C}_i\}_{i=1}^s$ be the distinct classes of $\Lambda(V)$ and define for each i : $V_i = \bigoplus_{\lambda \in \mathcal{C}_i} V_\lambda$. Clearly $V \cong \bigoplus_{i=1}^s V_i$ as T -modules, and moreover each summand is actually L -stable since W_L stabilizes each \mathcal{C}_i . Thus $V|_L \cong \bigoplus_{i=1}^s V_i$ and so $V|_L$ is decomposable whenever $s \geq 2$.

Now suppose $L \neq G$, so there is at least one positive root γ with U_γ not contained in L . Find an irreducible G -module $S(\mu)$, where $\Lambda(S(\mu))$ includes the weight $\mu - \gamma$. For example, if $\Lambda(S(\mu))$ is the saturated set of weights with highest weight μ and if $\mu - \gamma$ is dominant, then $\mu - \gamma \in \Lambda(S(\mu))$. In characteristic zero, any irreducible with $\mu - \gamma \in \Lambda_+$ will suffice, while in characteristic p , the Steinberg module $ST(p)$ or one of its higher versions $ST(p^r)$ will suffice.

Now if $\mu \sim (\mu - \gamma)$, then $\gamma \in Z\Phi_L$; but γ is a root of G , so $\gamma \in \Phi \cap Z\Phi_L = \Phi_L$. This is a contradiction to the choice of γ . Thus μ and $\mu - \gamma$ belong to different classes and so $s \geq 2$ in the above decomposition. ■

LEMMA 3.2. *Let H be a closed subgroup of G , and let $x \in G$. Then restriction $(-)|_H: \text{RAT}(G) \rightarrow \text{RAT}(H)$ is a topheavy functor iff restriction $(-)|_{H^x}: \text{RAT}(G) \rightarrow \text{RAT}(H^x)$ is topheavy, where $H^x = x^{-1}Hx$.*

Proof. Left for the reader. ■

Now let H be a closed connected subgroup containing T , with Levi decomposition $H = L \cdot U_1$. Assume that restriction $(-)|_H$ is a topheavy functor. If U_1 is trivial then $H = L$ is reductive, so $H = G$ by Lemma 3.1. So we may assume that U_1 is not trivial and apply [10, Sect. 30.3, Corollary A]. We obtain a parabolic subgroup P with $N_G(U_1) \subseteq P$ and $U_1 \subseteq R_u(P)$ (the unipotent radical of P). Moreover, P is a proper subgroup of G since U_1 is nontrivial. Now there is an $x \in G$ such that P^x is a standard

parabolic subgroup P_I with Levi decomposition $P_I = L_I \cdot U_I$, so $U_I = R_u(P)^x$. Thus $U_1^x \subseteq U_I$ and $H \subseteq N_G(U_1) \subseteq P$ so $H^x \subseteq P_I$. But H is parabolic iff H^x is, and moreover Lemma 3.2 implies that $(-)|_{H^x}$ is also topheavy. Thus we may replace H by H^x without loss of generality to assume that $H \subseteq P_I$ and $U_1 \subseteq U_I$. Now L is a reductive subgroup of P_I , so lives in a maximal reductive subgroup, which is a Levi factor of P_I , so we may also assume that $L \subseteq L_I$.

LEMMA 3.3. *Let $H \subseteq P_I$, $U_1 \subseteq U_I$, and $L \subseteq L_I$ and suppose that restriction $(-)|_H: \text{RAT}(G) \rightarrow \text{RAT}(H)$ is a topheavy functor. Then restriction $(-)|_L: \text{RAT}(L_I) \rightarrow \text{RAT}(L)$ is also topheavy.*

Proof. Let $S(\lambda)$ be an irreducible G -module and observe that $S(\lambda)|_{P_I}$ has socle $S_I(\lambda)$ by [14, Proposition 5.1] and Proposition 1. Consider $S_I(\lambda)|_L$ and suppose that it has a decomposable socle. So there are at least two irreducible L -modules S_1 and S_2 with $S_1 \oplus S_2 \subseteq S_I(\lambda)|_L$. But everything in U_I acts trivially on $S_I(\lambda)$ as the latter is an irreducible P_I -module. Thus $U_1 \subseteq U_I$ acts trivially on $S_I(\lambda)|_L$ and so the L -submodule $S_1 \oplus S_2$ automatically extends to an H -submodule of $S_I(\lambda)$ which in turn is an H -submodule of $S(\lambda)$ (as $H \subseteq P_I$). But then $S(\lambda)|_H$ is not a topheavy module, which is a contradiction.

Hence $S_I(\lambda)|_L$ is topheavy for all dominant λ . An arbitrary irreducible for L_I has highest weight $\mu \in A'_+$. But μ can always be written as a sum $\varepsilon + \lambda$, where ε is a character of L_I and $\lambda \in A_+$ (see [14, Sect. 3]). If $S_1 \oplus S_2 \subseteq S_I(\mu)|_L$, then as $S_I(\mu) \cong S_I(\lambda) \otimes \varepsilon$, we obtain $(S_1 \otimes -\varepsilon) \oplus (S_2 \otimes -\varepsilon) \subseteq S_I(\lambda)|_L$. But $S_i \otimes -\varepsilon$ is an irreducible for L , since a character of L_I is also one of L . Since λ is dominant, this is a contradiction, so $S_I(\mu)|_L$ is topheavy for every irreducible L_I -module $S_I(\mu)$ with $\mu \in A'_+$. ■

LEMMA 3.4. *Suppose H is closed connected subgroup of G such that $L_I \subseteq H \subseteq P_I$, where L_I is the common Levi factor of both H and P_I . Then restriction $(-)|_H: \text{RAT}(G) \rightarrow \text{RAT}(H)$ is a topheavy functor iff $H = P_I$.*

Proof. Let λ be a dominant character of T . Observe that

$$\begin{aligned} \text{Hom}_H(S_I(\mu), S(\lambda)) &\cong (\text{Hom}(S_I(\mu), S(\lambda))^{U_1})^{L_I} \\ &\cong (\text{Hom}(M(-\lambda), M_I(-\mu))^{U_1})^{L_I} \\ &\cong (\text{Hom}(M(-\lambda), k)^{U_1} \otimes M_I(-\mu))^{L_I}, \end{aligned}$$

because $U_1 \subseteq U_I$ acts trivially on $M_I(-\mu)$. This further reduces to

$$\begin{aligned} (S(\lambda)^{U_1} \otimes M_I(-\mu))^{L_I} &\cong (\text{Hom}(S_I(\mu), k) \otimes S(\lambda)^{U_1})^{L_I} \\ &\cong \text{Hom}(S_I(\mu), S(\lambda)^{U_1})^{L_I} \cong \text{Hom}_{L_I}(S_I(\mu), S(\lambda)^{U_1}). \end{aligned}$$

Now since $U_1 \subseteq U_I$ we have $S(\lambda)^{U_I} \subseteq S(\lambda)^{U_1}$ as an L_I -submodule, and $S(\lambda)^{U_1} \cong S_I(\lambda)$ by [12]. That is, we have $\text{Hom}_{L_I}(S_I(\lambda), S(\lambda)^{U_1}) \neq 0$ and the above string of isomorphisms shows that $\text{Hom}_H(S_I(\lambda), S(\lambda)|_H) \neq 0$. Since $(-)|_H$ is topheavy, this yields that $\text{Hom}_H(S_I(\lambda), S(\lambda)|_H)$ is one-dimensional and $\text{Hom}_H(S_I(\mu), S(\lambda)|_H) = 0$ if $\mu \neq \lambda$, and similarly for $\text{Hom}_{L_I}(S_I(\mu), S(\lambda)^{U_1})$.

Next, let S' be the subspace of $S(\lambda)$ spanned by all weight vectors with weights μ in the class of λ , under the equivalence relation $\mu \sim \lambda$ if $\lambda - \mu \in Z\Phi_I$. In [12], it is shown that $S_I(\lambda) \cong S(\lambda)^{U_I} = S'$, which is a direct summand of $S(\lambda)|_{L_I}$ as in Lemma 3.1.

Since $\dim \text{Hom}_{L_I}(S_I(\mu), S(\lambda)^{U_1}) = \delta_{\mu\lambda}$ (Kronecker delta), we see $S(\lambda)^{U_1}$ is a topheavy L_I -module. Since $S' \cap S(\lambda)^{U_1} \neq \phi$ (they both contain $S_I(\lambda)$), $S(\lambda)^{U_1}$ must be entirely contained in the L_I -summand S' (or else it would not be topheavy). But by [12], this summand consists only of the irreducible $S_I(\lambda)$. Thus if $(-)|_H$ is topheavy, we obtain $S(\lambda)^{U_1} = S(\lambda)^{U_I} \cong S_I(\lambda)$.

If H contained each root group U_α for $\alpha \in \Delta$, then H contains B , the subgroup generated by T and all these root groups. Hence $H = P_I$ and we are done. Otherwise there is a simple root α with U_α not contained in H . Suppose λ is a dominant character with $\lambda - \alpha$ also dominant and such that $S(\lambda)_{\lambda - \alpha} \neq 0$. Then any vector in this weight space spans an $H \cap B$ -stable line in $S(\lambda)$ of weight $\mu = \lambda - \alpha$. Thus

$$\begin{aligned} 0 \neq \text{Hom}_{H \cap B}(\mu, S(\lambda)) &\cong (\text{Hom}(M(-\lambda), -\mu)^{U_1})^{B \cap L_I} \\ &\cong (\text{Hom}(M(-\lambda), k)^{U_1} \otimes -\mu)^{B \cap L_I} \\ &\cong (S(\lambda)^{U_1} \otimes -\mu)^{B \cap L_I} \\ &\cong \text{Hom}_{B \cap L_I}(\mu, S(\lambda)^{U_1}). \end{aligned}$$

But $S(\lambda)^{U_1} \cong S_I(\lambda)$ so $0 \neq \text{Hom}_{B \cap L_I}(\mu, S_I(\lambda))$, a contradiction because the only $B \cap L_I$ -stable line in $S_I(\lambda)$ is the one spanned by a maximal vector of highest weight λ .

In characteristic zero, any irreducible $S(\lambda)$ with $\lambda - \alpha \in \Delta_+$ suffices while in prime characteristics, the Steinberg modules again do the job. It follows that $U_\alpha \subseteq H$ for all simple roots $\alpha \in \Delta$ so $H = P_I$ and we are done. ■

THEOREM 3.5. *Let H be a closed connected subgroup of G containing T . Then H is parabolic iff the restriction functor $(-)|_H: \text{RAT}(G) \rightarrow \text{RAT}(H)$ is a topheavy functor.*

Proof. As mentioned above, we may assume that $H \subseteq P_I$ with $L \subseteq L_I$ and $U_1 \subseteq U_I$. By virtue of Lemma 3.3, if $(-)|_H$ is topheavy, so is the functor $(-)|_L: \text{RAT}(L_I) \rightarrow \text{RAT}(L)$. But then $L = L_I$ by Lemma 3.1 applied to L_I .

Hence the Levi decomposition for H is $L_I \cdot U_1$ with $U_1 \subseteq U_I$ so $L_I \subseteq H \subseteq P_I$. Then $H = P_I$ by Lemma 3.4. Conversely if H is parabolic, then restriction is a topheavy functor by Proposition 1 together with [14, Proposition 5.1]. ■

4. TOPHEAVY INDUCTION FUNCTORS

Suppose H is a closed connected subgroup of G containing T , and assume that for each dominant character λ of T , $M(-\lambda) \subseteq M_L(-\lambda)|_H^G$. Then it follows from Theorem 3.5 and Proposition 1 that H is parabolic iff induction $(-)|_H^G$ is a topheavy functor which kills any $M_L(-\mu)$ if μ is not dominant.

Can we drop the hypothesis that $(-)|_H^G$ kills $M_L(-\mu)$ for nondominant μ 's? In other words, is H parabolic iff $(-)|_H^G$ is topheavy? We will see that the answer is frequently yes.

The first situation where an affirmative answer holds is the analogue of Lemma 3.4, where we make the additional hypothesis on H that $L_I \subseteq H \subseteq P_I$ for some $I \subseteq \Delta$. Define $\Phi_H = \{\gamma \in \Phi \mid U_\gamma \subseteq H\}$ and $\Delta_H = \Delta \cap \Phi_H$. Note that $I \subseteq \Delta_H$, and if J is another subset of Δ with $I \subseteq J \subseteq \Delta_H$, then $H \cap L_J = P_I \cap L_J$. Indeed if $x = lu \in L_I U_I = P_I$, then as $l \in L_I \subseteq L_J$, we see that $x \in L_J$ iff $u \in L_J$. Suppose $x \in P_I \cap L_J$, whence $u \in L_J \cap U_I$ so is a product of elements belonging to root groups U_γ for $\gamma \in \Phi_J^+ \subseteq \Phi_{\Delta_H}^+$. But $\Phi_{\Delta_H}^+ \subseteq \Phi_H$ so $u \in H$. Moreover $l \in L_I \subseteq H$, so $x \in H$ and this shows that $P_I \cap L_J \subseteq H \cap L_J$. But $H \subseteq P_I$, so $H \cap L_J \subseteq P_I \cap L_J$ also.

PROPOSITION 4.1. *Let H be a closed connected subgroup of G such that $L_I \subseteq H \subseteq P_I$. Let J be a set of simple roots with $I \subseteq J \subseteq \Delta_H$. Recall that an irreducible H -module is of the form $M_I(-\lambda)$ for some character $\lambda \in A_+^I$. Then:*

- (a) $M_I(-\lambda)|_H^{P_I} \neq 0$ iff $\lambda \in A_+^J$.
- (b) $M_I(-\lambda)|_H^{P_I} \neq 0 \Rightarrow \text{Soc}(M_I(-\lambda)|_H^{P_I}) \cong M_J(-\lambda)$.

In particular $J \subseteq \Delta_H \Rightarrow (-)|_H^{P_I}$ is a topheavy functor.

Proof. An arbitrary irreducible P_J -module has the form $M_J(-\mu)$ for some $\mu \in A_+^J$. Apply reciprocity, transitivity, and the tensor identity to obtain $\text{Hom}_{P_J}(M_J(-\mu), M_I(-\lambda)|_H^{P_I}) \cong \text{Hom}_{P_I}(M_J(-\mu), M_I(-\lambda)|_H^{P_I}) \cong \text{Hom}_{P_I}(M_J(-\mu), k|_H^{P_I} \otimes M_I(-\lambda)) \cong [\text{Hom}(M_J(-\mu), k|_H^{P_I} \otimes M_I(-\lambda))^{U_I}]^{L_I} \cong [\text{Hom}(M_J(-\mu), k|_H^{P_I})^{U_I} \otimes M_I(-\lambda)]^{L_I}$, the last isomorphism because U_I

acts trivially on $M_I(-\lambda)$. To reduce this further, note that $k|_{H^I}^P|_{U_I} \cong k|_{H \cap U_I}^{U_I}$ by [6, 4.1] so we obtain

$$\begin{aligned} \text{Hom}_{P_J}(M_J(-\mu), M_I(-\lambda)|_{H^I}^P) &\cong [\text{Hom}_{U_I}(M_J(-\mu), k|_{H \cap U_I}^{U_I}) \otimes M_I(-\lambda)]^{L_I} \\ &\cong [\text{Hom}_{H \cap U_I}(M_J(-\mu), k) \otimes M_I(-\lambda)]^{L_I} \\ &\cong [(M_J(-\mu))^*]^{H \cap U_I} \otimes M_I(-\lambda)]^{L_I} \\ &\cong [S_J(\mu)^{H \cap U_I} \otimes M_I(-\lambda)]^{L_I}. \end{aligned}$$

But $U_I \cong U_{J,I} \cdot U_J$ (notation as in Section 2) and $J \subseteq \Delta_H$ so $H \cap U_I$ contains $U_{J,I}$. Thus $H \cap U_I \cong (H \cap U_J) \cdot U_{J,I}$ and $S_J(\mu)^{H \cap U_I} \cong (S_J(\mu)^{H \cap U_J})^{U_{J,I}} \cong S_J(\mu)^{U_{J,I}}$ because U_J acts trivially on $S_J(\mu)$.

But $S_J(\mu)^{U_{J,I}} \cong S_I(\mu)$ by Lemma 2.1. Combined with the above, this yields

$$\begin{aligned} \text{Hom}_{P_J}(M_J(-\mu), M_I(-\lambda)|_{H^I}^P) &\cong (S_I(\mu) \otimes M_I(-\lambda))^{L_I} \\ &\cong \text{Hom}_{L_I}(M_I(-\mu), M_I(-\lambda)), \end{aligned}$$

which is zero if $\lambda \neq \mu$ and one-dimensional if $\lambda = \mu$ by Schur's lemma. This proves (b), and (a) as well since $\lambda = \mu \Rightarrow \lambda \in A_+^J$. ■

Now consider induction $(-)|_{H^I}^P$ when J is not contained in Δ_H . Theorem 2.2 shows that the above theorem can fail in this case. In fact it must fail, as Theorem 4.1 has the following converse:

THEOREM 4.2. *Let H be a connected subgroup of G with $L_I \subseteq H \subseteq P_I$. Suppose that $\text{Soc}(M_I(-\lambda)|_{H^I}^P) \cong M_J(-\lambda)$ for all dominant characters λ . Then $J \subseteq \Delta_H$.*

Proof. First consider the case when I is empty, so $T \subseteq H \subseteq B$ and H is solvable. An irreducible H -module has the form λ . Suppose that J is not contained in Δ_H , so there exists a simple root $\alpha \in J$ with U_α not contained in H . In particular $H \subseteq T \cdot U_{\{\alpha\}}$, where $U_{\{\alpha\}}$ is the unipotent radical of the minimal parabolic P_α . So the elements of $k[P_\alpha] \otimes -\lambda$ which are fixed by $T \cdot U_{\{\alpha\}}$ are automatically fixed by H . Then $k[P_\alpha] \otimes_{T \cdot U_{\{\alpha\}}} -\lambda \subseteq k[P_\alpha] \otimes_H -\lambda$; that is $-\lambda|_{T \cdot U_{\{\alpha\}}}^P \subseteq -\lambda|_H^P$ (see [5, Remark 1.3]).

By Theorem 2.2(b), there are dominant characters λ for which $-\lambda|_{T \cdot U_{\{\alpha\}}}^P$ has a decomposable socle. Indeed we can arrange things so that λ and $\lambda + \alpha$ are both dominant and so that $M_{\{\alpha\}}(-(\lambda + \alpha)) \subseteq \text{Soc}(-\lambda|_{T \cdot U_{\{\alpha\}}}^P)$. As $-\lambda|_{T \cdot U_{\{\alpha\}}}^P \subseteq -\lambda|_H^P$ we see $M_{\{\alpha\}}(-(\lambda + \alpha)) \subseteq \text{Soc}(-\lambda|_H^P)$ also. Now apply induction $(-)|_{P_\alpha}^P$ and note that $\lambda + \alpha \in A_+ \subseteq A_+^J$, so $M_{\{\alpha\}}(-(\lambda + \alpha))|_{P_\alpha}^P$ is nonzero and contains $M_J(-(\lambda + \alpha))$ as a submodule by [14, 5.1].

We obtain inclusions $M_J(-(\lambda + \alpha)) \subseteq M_{\{\alpha\}}(-(\lambda + \alpha))|_{P_\alpha}^P \subseteq -\lambda|_H^P$ because induction is left exact and transitive. The same argument yields

inclusions $M_J(-\lambda) \subseteq M_{\{x\}}(-\lambda)|_{P_x}^{P_J} \subseteq -\lambda|_H^{P_J}$, so $\text{Soc}(-\lambda|_H^{P_J})$ contains both $M_J(-\lambda)$ and $M_J(-(\lambda + \alpha))$. This is a contradiction so there cannot exist any such simple roots α , so $J \subseteq \Delta_H$ after all.

Now suppose that I is nonempty. Let λ be a dominant weight such that $-\lambda|_B^{P_I}$ is irreducible, isomorphic to $M_I(-\lambda)$. Then observe that

$$\begin{aligned} M_I(-\lambda)|_H^{P_I} &\cong -\lambda|_B^{P_I}|_H^{P_I} \cong -\lambda|_B^{P_I}|_H^{P_I}|_{P_I}^{P_I} \\ &\cong [(k|_H^{P_I}) \otimes (-\lambda|_B^{P_I})]|_{P_I}^{P_I} \cong [(k|_H^{P_I}|_B \otimes -\lambda)|_B^{P_I}]|_{P_I}^{P_I} \end{aligned}$$

by transitivity and two uses of the tensor identity. But $HB = P_I$, so [6, 4.1] together with transitivity and the tensor identity again gives that

$$M_I(-\lambda)|_H^{P_I} \cong (k|_H^{P_I}|_B \otimes -\lambda)|_B^{P_I} \cong (k|_{H \cap B}^B \otimes -\lambda)|_B^{P_I} \cong -\lambda|_{H \cap B}^{P_I}.$$

Moreover $H \cap B$ is a connected solvable group with $T \subseteq H \cap B \subseteq B$ and with $\Delta_{H \cap B} = \Delta_H$. The isomorphism $M_I(-\lambda)|_H^{P_I} \cong -\lambda|_{H \cap B}^{P_I}$ shows that $-\lambda|_{H \cap B}^{P_I}$ is topheavy if $M_I(-\lambda)|_H^{P_I}$ is. But if $\Delta_{H \cap B}$ does not contain J , then by the solvable case we can find dominant characters λ for which $-\lambda|_H^{P_I}$ is not topheavy. Thus we obtain a contradiction to the hypothesis that $M_I(-\lambda)|_H^{P_I}$ be a topheavy module for all dominant characters, hence $J \subseteq \Delta_{H \cap B} = \Delta_H$ as claimed.

It remains to see that there always exist dominant characters λ such that $-\lambda|_B^{P_I}$ is irreducible. If the characteristic of k is zero, this is always true, while in prime characteristics, it holds for the Steinberg characters $\lambda_r = (p^r - 1)\rho$. ■

COROLLARY 4.3. *Let H be a connected subgroup of G with $L_I \subseteq H \subseteq P_I$.*

(a) *If $(-)|_H^{P_I}$ is a topheavy functor, then $H \cap L_J$ is a parabolic subgroup of L_J .*

(b) *In particular $(-)|_H^G$ is topheavy iff $H = P_I$.*

Proof. By Theorem 4.2 we know that $J \subseteq \Delta_H$, so $H \cap L_J = P_I \cap L_J$ as remarked prior to Proposition 4.1. This shows (a), and (b) follows by taking the case $J = \Delta$. ■

Corollary 4.3 is the analogue of Lemma 3.4 for the induction functor. Having come this far, one would hope to be able to reduce the general case for a topheavy induction functor to the case $L_I \subseteq H \subseteq P_I$, following the steps we used in Section 3 for the restriction functor. However, at this stage it is unclear how to prove the analogue of Lemma 3.1 without adding the extra hypothesis that induction kills any $M_L(-\lambda)$ when λ is not dominant. (As we know from Proposition 1, this is tantamount to assuming that restriction is topheavy.)

For example, consider the case when G is of type G_2 and let H be the reductive subgroup consisting of T together with all root groups U_α for α a long root. Then H is a reductive group of type A_2 . Since H is not all of G , we know restriction is not topheavy. Let $\Delta = \{\alpha, \beta\}$, where α is the short root. Then the positive root $\beta + 2\alpha$ is in fact dominant for G_2 and the reader should have no trouble using the technique of Lemma 3.1 to decompose $S(\beta + 2\alpha)|_{A_2}$ into a direct sum of two three-dimensional irreducibles for A_2 . One has highest weight $\beta + 2\alpha$ and one has highest weight $\beta + \alpha$, which is a dominant weight for A_2 (using the base $\Delta = \{\beta, \beta + 3\alpha\}$) but not dominant for G_2 .

In this case, although restriction is not topheavy, it is possible (though unlikely) that induction may be topheavy. Proposition 1 will not be violated, as induction does not kill the H -irreducibles whose lowest weights are not negative dominant. Indeed in the example at hand both of the irreducibles for A_2 induce up to the same module for G_2 , although one has a lowest weight which is not negative dominant for G_2 .

COROLLARY 4.4. *Let H be a solvable connected subgroup of G containing T . Then induction $(-)|_H^G$ is topheavy iff H is a Borel subgroup.*

Proof. Since H is solvable, there is a Borel subgroup B with $T \subseteq H \subseteq B$. The result follows from Corollary 4.3 with I taken to be empty. ■

COROLLARY 4.5. *Suppose $\text{char } k = 0$. If H is a connected subgroup of G containing T , then H is parabolic iff induction $(-)|_H^G$ is topheavy.*

Proof. If H is parabolic we know that induction is topheavy. Conversely, let $(-)|_H^G$ be topheavy. Let $L \cdot U_1$ be the Levi decomposition of H so an irreducible H -module, denoted $M_L(-\lambda)$, has lowest weight $-\lambda$, where λ is dominant for L 's root system Φ_L . Choose a Borel subgroup B containing both T and U_1 . Then $H \cap B = L \cdot U_1 \cap B = (B \cap L) \cdot U_1$ and $B/(H \cap B) \cong L \cdot U_1 / ((B \cap L) \cdot U_1) \cong L / (B \cap L)$ is projective because $B \cap L$ is a Borel subgroup of L .

Observe that $L(H \cap B) = H$, so [6, 4.1] implies that $-\lambda|_{H \cap B}^H \cong -\lambda|_{B \cap L}^L$ for all λ , which is 0 if λ is not dominant for L . Because $\text{char } k = 0$, $-\lambda|_{B \cap L}^L$ is the L -irreducible $M_L(-\lambda)$ with lowest weight $-\lambda$, hence $-\lambda|_{H \cap B}^H \cong M_L(-\lambda)$ as H -modules. Thus $M_L(-\lambda)|_H^G \cong -\lambda|_{H \cap B}^G$ by transitivity. Since $(-)|_H^G$ is topheavy, we see $-\lambda|_{H \cap B}^G$ has an irreducible socle $M(-\lambda)$ at least for all dominant λ , which is enough to force $\Delta \subseteq \Delta_{H \cap B}$ by Theorem 4.2. Hence $B = H \cap B$ and so $B \subseteq H$. But then H is parabolic because it contains a Borel subgroup of G . ■

In particular, if $\text{char } k = 0$, this shows that in our example above with $A_2 \subseteq G_2$, induction is not topheavy. We expect that Corollary 4.5 is true in

all characteristics, but we do not have a way to prove it yet. Of course, we could state a result which reads that in all characteristics, H is parabolic iff there is a Borel subgroup B such that induction $(-)|_{H \cap B}^G$ is topheavy, but there does not seem to be any real advantage to this.

APPENDIX: APPLICATIONS TO COUPLED PARABOLIC SYSTEMS

In this section G is a connected semisimple group and H is a closed subgroup. If $V \in \text{RAT}(H)$ and $x \in G$ then V^x denotes the rational module for $H^x = x^{-1}Hx$ obtained from V via conjugation by x . If $\lambda \in \Lambda$, let $\lambda^* = -w_0(\lambda)$, where w_0 is the long word in G 's Weyl group W . If J is a set of weights then let $J^* = \{\lambda^* \mid \lambda \in J\}$.

In [6] the authors prove a very general ‘‘Mackey decomposition theorem’’ which has a version for the n th derived functors of induction provided n is small. When the subgroups involved are parabolic, this theorem specializes (see [6, 4.5]) to the following:

THEOREM 5.1. *Let G be as above, with root system Φ and simple roots Δ . Let J and K be proper subsets of Δ such that $J^* \cup K = \Delta$. Then for all rational P_J -modules V there is an isomorphism of rational P_K -modules: $V|_{P_J}^G|_{P_K} \cong V^{w_0}|_H|^{P_K}$, where $H = P_J^{w_0} \cap P_K$.*

Moreover, in [14] it is shown that Theorem 5.1 extends to $L_{P_J, G}^n(V)|_{P_K}$ for n bounded above by an easily computed integer which depends only on J and K . The proof of the more general [6, Theorem 4.4] requires the sophisticated techniques of group-schemes. However, for the purposes of proving Theorem 5.1, where the subgroups are parabolic and $n = 0$, we may take a more elementary approach.

Proof of Theorem 5.1. Let $H_K^J = P_J^{w_0} \cap P_K$. First observe that the subvariety $P_K P_J^{w_0}$ of G is isomorphic to the quotient of the product variety $P_K \times P_J^{w_0}$ by the subvariety $H = \{(h, h^{-1}) \mid h \in H_K^J\}$. Indeed multiplication $m: P_K \times P_J^{w_0} \rightarrow G$ has image $P_K P_J^{w_0}$ and fibers isomorphic to H . It follows for the coordinate rings that $k[P_K P_J^{w_0}] \cong k[(P_K \times P_J^{w_0})/H] \cong k[P_K \times P_J^{w_0}]^H$. But the coordinate ring of a direct product is the tensor product of the coordinate rings, so this last space identifies naturally with the elements of $k[P_K] \otimes k[P_J^{w_0}]$ which are fixed under the action of H_K^J given by $f \otimes g \rightarrow f \cdot h^{-1} \otimes h \cdot g$. Thus we have an isomorphism of k -algebras $\Theta: k[P_K P_J^{w_0}] \cong k[P_K] \otimes_H k[P_J^{w_0}]$, where we have abbreviated H_K^J by H .

As this isomorphism is induced by the multiplication map m , it is clear that it respects the action of P_K on the left and of $P_J^{w_0}$ on the right, so Θ is also a $(P_K, P_J^{w_0})$ -bimodule map. We remark that for any rational

H -module V and any overgroup P of H , then $V|_H^P$ identifies naturally with $k[P] \otimes_H V$ (see [5, Remark 1.3]).

Next, $P_K P_J^{w_0}$ is dense in G (it contains a translate of the big cell) so it follows that the restriction map $r: k[G] \rightarrow k[P_K P_J^{w_0}]$ is injective. Moreover, we claim that $\text{codim}_G(G - P_K P_J^{w_0}) \geq 2$, so it follows from [7, p. 239, Lemma 1] that r is surjective also. Assuming this for the moment, we obtain an isomorphism of left modules,

$$k[G]|_{P_K} \cong k[P_K] \otimes_H k[P_J^{w_0}],$$

because r also respects the action of P_K . Now if $V \in \text{RAT}(P_J)$, then

$$\begin{aligned} V|_{P_J}^G|_{P_K} &\cong V^{w_0}|_{P_J^{w_0}}^G|_{P_K} \cong (k[G] \otimes_{P_J^{w_0}} V^{w_0})|_{P_K} \cong k[G]|_{P_K} \otimes_{P_J^{w_0}} V^{w_0} \\ &\cong k[P_K] \otimes_H k[P_J^{w_0}] \otimes_{P_J^{w_0}} V^{w_0} \cong k[P_K] \otimes_H (V^{w_0}|_{P_J^{w_0}}^{P_J^{w_0}}) \\ &\cong k[P_K] \otimes_H V^{w_0} \cong V^{w_0}|_H|_{P_K}. \end{aligned}$$

It remains to show $\text{codim}_G(G - P_K P_J^{w_0}) \geq 2$. But this codimension is equal to $\dim G - \dim(G - P_K P_J^{w_0})$, and translating by w_0 does not affect the dimension, so this further reduces to $\dim G - \dim(G - P_K w_0 P_J)$. But the Bruhat decomposition shows that

$$P_K w_0 P_J = \bigcup_{w \in W_K w_0 W_J} BwB = \bigcup_{w \in W_K W_{J^*}} Bw w_0 B.$$

Hence $G - P_K w_0 P_J = \bigcup Bw w_0 B$, where this last union runs over all w not in $W_K W_{J^*}$. But $\dim(G - P_K w_0 P_J)$ is the dimension of its largest double coset. Since $\dim BwB = \dim B + l(w)$ we obtain that $\dim(G - P_K w_0 P_J) = \dim B + \max\{l(w w_0) \mid w \text{ is not in } W_K W_{J^*}\}$. But $l(w w_0) = l(w_0) - l(w)$ (see [4]), so $l(w w_0)$ attains its maximum when $l(w)$ attains its minimum. Thus $\dim(G - P_K w_0 P_J) = \dim B + l(w_0) - \min\{l(w) \mid w \text{ is not in } W_K W_{J^*}\}$. Since $\dim G = \dim B + l(w_0)$, it follows that $\text{codim}_G(G - P_K w_0 P_J) = \min\{l(w) \mid w \text{ is not in } W_K W_{J^*}\}$. But $J^* \cup K = \Delta$ implies that each simple reflection s_α for $\alpha \in \Delta$ belongs to either W_K or W_{J^*} , so the minimum length of a word not in $W_K W_{J^*}$ is at least 2, as claimed. ■

In [14], besides extending Theorem 5.1 to derived functors, some basic properties of $(-)|_H^{P_K}$ were investigated, where $H = H_K^J = P_J^{w_0} \cap P_K$. Certainly such information is needed if one wishes to apply Theorem 5.1. Any subgroup of the form H_K^J we call a *coupled parabolic system*, or just a ‘‘CPS’’-subgroup of G . Assume $J^* \cup K = \Delta$. In [14], the Levi decomposition of H_K^J is computed and is given by

$$H_K^J = L_I \cdot (U_{J^*, I} \times U_{K, I}^-),$$

where $I = J^* \cap K$.

Then [14, Corollary 7.19] says that $(-)|_H^{P_K}$ is topheavy and [14, Theorem 7.20] says that $(-)|_H^{P_K}$ preserves finite dimensionality (see [13]). Moreover in certain cases when H is solvable and G is of type A_n or B_2 , the proof that $(-)|_H^{P_K}$ preserves finite dimensionality can be refined to give good filtrations of $\lambda|_H^{P_K}$ (see [14, Theorems 7.10, and 7.16]).

However, $(-)|_H^G$ need not be topheavy, and the derived functors $L_{H, P_K}^n(-)$ need not preserve finite dimensionality for $n > 0$, as discussed in [14, Sect. 7]. It was largely these results which raised the question of just when induction from a closed subgroup behaves in this way.

Observe that because H_K^J contains $U_{K, I}^-$, H_K^J is not imbedded in the standard parabolic P_I , although its Levi factor is L_I . We get around this by replacing B with $B_K = B^{w_K}$, where w_K is the long word in W_K . Thus, Δ is replaced by $\Delta' = \{w_K(\alpha) \mid \alpha \in \Delta\}$, and relative to this new simple system we have $-K \subseteq \Delta'$, and “ P_{-K} ” relative to B_K is just the original P_K relative to B . Then we have inclusions $L_I \subseteq H_K^J \subseteq P_{-I} \subseteq P_{-K} = P_K$, so we may now give an alternate proof of [14, Theorem 7.19].

PROPOSITION 5.2. *Let $H = H_K^J$ be a coupled parabolic system in G . Then induction $(-)|_H^{P_K}$ is a topheavy functor.*

Proof. H contains L_I , so in particular $-K \subseteq \Delta_H$ (relative to Δ'). The result follows from Proposition 4.1. ■

On the other hand, while $H_K^J \cap L_K$ is parabolic in L_K , H is certainly not parabolic in G , so Corollary 4.3 explains why $(-)|_H^G$ is not a topheavy functor.

Next, let G be $SL_3(k)$, and let $J = K = \{\alpha_2\} \subseteq \Delta = \{\alpha_1, \alpha_2\}$. Then $H_K^J \cong T \cdot (U_{\alpha_1} \times U_{-\alpha_2})$ is a solvable subgroup. Here w_K is the reflection s_{α_2} and $\Delta' = \{-\alpha_2, \alpha_1 + \alpha_2\}$. The matrices in H and B_K have the forms:

$$\begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix},$$

respectively.

Relative to B_K , $H = T \cdot U_{\{\beta\}}$, where β is the simple root $\alpha_1 + \alpha_2$ in Δ' . The results of [13] apply, so $(-)|_H^G$ preserves finite dimensionality, while $(-)|_H^{P_K}$ does not. It follows that $(-)|_H^{P_K}$ preserves finite dimensionality. Indeed we have an exact sequence similar to [13, 4.5.2],

$$0 \rightarrow \lambda|_{B_K}^{P_K} \rightarrow \lambda|_H^{P_K} \rightarrow Q|_{B_K}^{P_K} \rightarrow 0,$$

where $Q \cong (\lambda|_{B_K}^{B_K})/\lambda$. Apply $(-)|_H^{P_J^{y_0}}$ to this sequence to obtain $\lambda|_H^{P_J^{y_0}} \subseteq \lambda|_H^{P_K}|_H^{P_J^{y_0}}$, while the latter term is isomorphic to $k[P_J^{y_0}] \otimes_H k[P_K] \otimes_H \lambda$.

Similarly $\lambda|_H^{P_K} \subseteq \lambda|_H^{P_J^{w_0}}|_H^{P_K} \cong k[P_K] \otimes_H k[P_J^{w_0}] \otimes_H \lambda$. But $k[P_K] \otimes_H k[P_J^{w_0}] \cong k[G]|_{P_K}$ as in the proof of Theorem 5.1. Thus $\lambda|_H^{P_K} \subseteq (k[G] \otimes_H \lambda)|_{P_K} \cong \lambda|_H^G|_{P_K}$, and similarly $\lambda|_H^{P_J^{w_0}} \subseteq \lambda|_H^G|_{P_J^{w_0}}$ so in this case $(-)|_H^{P_K}$ and $(-)|_H^{P_J^{w_0}}$ preserve finite dimensionality because $(-)|_H^G$ does. Moreover the filtration of [14, Theorem 7.10] is likely to be an instance of the filtration of [13, Proposition 2.1d] on the dual module. So for $G = SL_3(k)$ the results of [14] may all be explained on the basis of the results of this paper and of [13].

More generally, if $G = SL_{n+1}(k)$, let $J^* = \{\alpha_1, \alpha_2, \dots, \alpha_j\}$ and let $K = \{\alpha_{j+1}, \dots, \alpha_n\}$. Then similar techniques show $(-)|_H^G$ preserves finite dimensionality. Indeed H contains $T \cdot U_{\Delta' - \{\gamma\}}$, where $\gamma = w_K(\alpha_n)$ is a simple root in Δ' . For the sake of illustration, let $n=6$ and $j=3$. Here the matrices for H and $T \cdot U_{\Delta' - \{\gamma\}}$ have the respective forms:

$$\begin{pmatrix} * & * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & 0 & * \end{pmatrix}.$$

Thus $(k[G] \otimes V)^H \subseteq (k[G] \otimes V)^{T \cdot U_{\Delta' - \{\gamma\}}}$ for $V \in \text{RAT}(H)$; that is to say $V|_H^G \subseteq V|_{T \cdot U_{\Delta' - \{\gamma\}}}^G$. It follows as above that $(-)|_H^{P_K}$ preserves finite dimensionality, which gives another proof of [14, Corollary 7.7]. Of course the proof of [14, Theorem 7.20] is more direct and more general.

This proof also shows that $(-)|_H^G$ preserves finite dimensionality. But H is not parabolic in G so if $\text{char } k = 0$ then the main result of [13] implies that there is some $n > 0$ for which $L_{H, G}^n(-)$ does not preserve finite dimensionality. In fact it can be shown that there is some n for which $L_{H, P_K}^n(-)$ does not preserve finite dimensionality, by considering the spectral sequence of induction obtained from the composite $(-)|_H^G = (-)|_{P_K}^G \circ (-)|_H^{P_K}$.

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