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A Cohomological Characterization of Parabolic Subgroups of Reductive Algebraic Groups

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1. INTRODUCTION

Let G be an algebraic group defined over an algebraically closed field k, and RAT(G) the category of rational G-modules. For any closed subgroup H of G and any rational H-module V, let $V|_H^G$ denote the rational G-module induced from V. The induction functor $(-)|_H^G\colon \mathrm{RAT}(H) \to \mathrm{RAT}(G)$ is left exact and we denote its derived functors by $L^n_{H,G}(-)$ for $n = 0, 1, 2, \ldots$.

Of course these derived functors are 0 for $n \geqslant 1$ whenever $(-)|_H^G$ is exact, and by a result of Cline, Parshall, and Scott [3] this happens precisely when G/H is an affine variety. So the case of an affine quotient is characterized by the vanishing of the higher derived functors. In a like vein, in this paper we will attempt to find a property of these functors which characterizes the case of a projective quotient. Since by definition G/H is projective iff $H = P$ is a parabolic subgroup of G, we are in effect looking for a cohomological characterization of parabolic subgroups of G.

The property of the functors $L^n_{H,G}(-)$ we consider is a certain finiteness condition explained below. The author was originally led to consider this question in [14], where induction from certain nonparabolic subgroups is studied and is shown to have this finiteness property.

It is well known that for every rational H-module V there is an induced bundle L_V on G/H with rank equal to dim V, such that $L^n_{H,G}(V)$ identifies naturally with the sheaf cohomology group $H^n((G/H, L_V)$. See [4] for details.

Thus we may apply standard results from algebraic geometry to obtain some basic facts. For example, $L^n_{H,G}(-) \equiv 0$ for $n > \dim(G/H)$, and Serre's theorem gives the easy half of the result of Cline, Parshall, and Scott men-

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tioned above. Moreover, if V is finite dimensional then L_V is a coherent sheaf. If G/P is projective then by a result of Serre $H^n(G/P, L_V)$ is finite dimensional for all n. In other words $L^n_{P,G}(V)$ is finite dimensional whenever V is. We shall say that $L^n_{P,G}(-)$ *preserves finite dimensionality*. These theorems of Serre and Grothendieck may be found in Chapter 3 of [8].

Unlike Serre's characterization of affine spaces, the projective version is not known to have a converse. Let G be a reductive group and H a closed subgroup (which we hope to prove is parabolic). For technical reasons it is easiest to work with subgroups H which are connected and contain some maximal torus T of G. (Certainly H must have those properties, if it is to be parabolic in G). We conjecture that the converse to Serre's result holds in these circumstances. That is, we have:

Conjecture A. Let G be a reductive algebraic group, and H a closed, connected subgroup of G containing T. Then H is parabolic iff $L^n_{H,G}(-)$ preserves finite dimensionality for all n.

In Section 2 we reduce the conjecture to the case when H is solvable. This part of the proof does not depend on char k. We also take a detailed look at some special cases. In Section 3 we handle the solvable case when char $k = 0$. In Section 4 we conclude with some partial results in the prime characteristic case and some related results.

We set up some notation which will hold throughout the remainder of the paper. The end (or absence) of a proof is indicated by the symbol ∎. Let Φ denote the root system of the pair (G, T), Λ the weight lattice, and let W be the Weyl group. The characters of T form a sublattice of Λ. Once a Borel subgroup B is fixed, it corresponds to a choice of simple roots Δ. Each character λ of T determines a one-dimensional B-module which we also denote by λ. We will use $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ to denote the set of fundamental dominant weights dual to Δ, Λ_+ for the set of dominant weights. More generally Λ^J_+ is $\{\lambda \in \Lambda \mid \langle \lambda, \gamma \rangle \geqslant 0 \forall \gamma \in J\}$, the set of "$J$-dominant" weights, where $J \subseteq \Delta$. Here $\langle \lambda, \gamma \rangle$ is an abbreviation for $2(\lambda, \gamma)/(\gamma, \gamma)$ as in [9].

If H is a closed, connected subgroup of G containing T, then the results of [1] imply that H has a Levi decomposition $H = L \cdot U_1$ (semidirect product) where U_1 is the unipotent radical of H and L is reductive.

If $V \in \mathrm{RAT}(H)$, then V^H is the fixed point space of V. If V is an irreducible H-module, then V^{U_1} is nonempty (because U_1 is unipotent) and a submodule (because $U_1 \lhd H$), hence is all of V making it the same thing as an irreducible L-module. Since $T \subseteq L$ and L is reductive we can list these irreducibles as they have highest weights which are dominant weights for the pair (L, T). For example, if H is the standard parabolic P_J, an irreducible P_J-module has its highest weight in Λ^J_+, and there is one

P_J-irreducible $S_J(\mu)$ with highest weight μ for each J-dominant character of T. See [14] for more details. In particular, $S(\mu)$ will denote the G-irreducible with highest weight $\mu \in \Lambda_+$.

Because $T \subseteq H$, every rational H-module is a direct sum of T-weight spaces, so one may define $\Lambda(V) = \{\lambda \in \Lambda \mid V_\lambda \neq 0\}$, the T-weights of V, and the formal character $\chi(V)$ of V which is the following formal sum in the integral group ring of Λ:

$$\chi(V) = \sum_{\lambda \in \Lambda(V)} m_\lambda e^\lambda,$$

where $m_\lambda = \dim V_\lambda$.

Recall that the unipotent radical U of B is (as an algebraic variety) isomorphic to a direct product of one parameter root groups U_γ, as γ ranges over Φ^+. In particular, if $k[X_\gamma]$ denotes the polynomial ring which is the coordinate ring of U_γ, then as k-algebras, $k[U] \cong \bigotimes_{\gamma \in \Phi^+} k[X_\gamma]$. Moreover, as T normalizes each U_γ, this decomposition is T-equivariant, so we can consider the formal character of $k[U]$. In this decomposition, the monomial X_γ has T-weight $-\gamma$.

Clearly, the same holds when instead of U, one works with the variety $\prod U_\gamma$ where the product is over any subset S of Φ^+, even if this variety is not a subgroup. In particular if S is any subset of Φ^+, let A_S denote the k-algebra which is the coordinate ring of this variety; so $A_S = \bigotimes_{\gamma \in S} k[X_\gamma]$.

The formal character of A_S is

$$\begin{aligned}
\chi(A_S) = \chi\left(\bigotimes_{\gamma \in S} k[X_\gamma]\right) &= \prod_{\gamma \in S} \chi(k[X_\gamma]) \\
&= \prod_{\gamma \in S} (1 + e^{-\gamma} + e^{-2\gamma} + \cdots) \\
&= \prod_{\gamma \in S} (1 + e^{-\gamma} + (e^{-\gamma})^2 + \cdots) \\
&= \prod_{\gamma \in S} (1 - e^{-\gamma})^{-1}.
\end{aligned}$$

If $J \subseteq \Delta$, then let E_J be the subspace of E spanned by J where E is the ambient Euclidean space of Φ. Let $L_J U_J$ be the Levi decomposition of P_J, so L_J is reductive with root system Φ_J and Weyl group W_J. We let $Q = Z\Phi$ be the root lattice and $Q_J = ZJ$ the root lattice in E_J (where Z denotes the integers). Also U_J is a product (as varieties) of root groups U_γ for $\gamma \in \Phi^+ - \Phi_J^+$ where Φ_J^+ are the positive roots in Φ_J. More generally let $U_{K,J}$ denote the unipotent radical of $P_J \cap L_K$ if $J \subseteq K \subseteq \Delta$ which is a product of the U_γ for $\gamma \in \Phi_K^+ - \Phi_J^+$. Similarly U_J^- denotes the product of the U_γ for $-\gamma \in \Phi^+ - \Phi_J^+$.

If S is a finite set of weights, let $\Lambda_+(S)$ denote $\{\sum_{\gamma \in S} n_\gamma \gamma \mid n_\gamma$ is a non-negative integer $\forall \gamma \in S\}$ which we call the integral cone spanned by S. For example, $\Lambda_+(\Omega)$ is just Λ_+, while $\Lambda_+(\Delta)$ is the positive part Q^+ of the root lattice Q. Similarly we have $Q_J^+ = \Lambda_+(J)$. Let $l(w)$ denote the length of a reduced expression for $w \in W$. We define the "dot" action of W by $w \cdot \lambda = w(\lambda - \rho) + \rho$, where $\rho = \sum_{i=1}^n \omega_i$.

We conclude this section with the following remarks about the functors $L^n_{P,G}(-)$ where P is a parabolic subgroup of a reductive group G. In characteristic 0, the Borel–Weil–Bott theorem completely describes their effects on any irreducible P-module. We will find this theorem useful in Section 3 so we state it in the case of a Borel subgroup B.

THEOREM 1 (Bott [2]). *Suppose* char $k = 0$ *and let λ be a character of T, regarded as a B-module. Then*:

(a) *If there is an $\alpha \in \Phi^+$ with $\langle \lambda, \alpha \rangle = +1$, then $L^n_{B,G}(\lambda) = 0$ for all n.*

(b) *Otherwise there is a unique $w \in W$ with $w \cdot \lambda \in -\Lambda_+$. Then $L^n_{B,G}(\lambda) = 0$ unless $n = l(w)$ in which case it is isomorphic to $w \cdot \lambda \mid^G_B$, which is an irreducible G-module with lowest weight $w \cdot \lambda$.* ∎

In prime characteristics, only part of this remains true (see Kempf [10]), but it is enough to show that $L^n_{P,G}(-)$ preserves finite dimensionality without using Serre's result. We illustrate this in Section 2.

2. SOME SPECIAL CASES

Now, assume $T \subseteq H \subseteq G$, and let $L \cdot U_1$ be the Levi decomposition of H. Choose any Borel subgroup B with $T \subseteq B$ and $U_1 \subseteq B$. Then $B \cap H = (B \cap L) \cdot U_1$. Note that $B \cap H$ is also a closed connected subgroup containing T, which is solvable (since $B \cap H \subseteq B$). Thus, an irreducible $B \cap H$-module is of the form λ, for some $\lambda \in \Lambda$. Observe that $H/(B \cap H) \cong L \cdot U_1/(B \cap L) \cdot U_1 \cong L/(B \cap L)$ which is projective because $B \cap L$ is a Borel subroup of L. In particular $L^q_{B \cap H,H}(-)$ preserves finite dimensionality for all q. Assume that $L^p_{H,G}(-)$ also preserves finite dimensionality for all p. It follows that the composite $L^p_{H,G}(L^q_{B \cap H,H}(-))$ preserves finite dimensionality for all p, q.

But this is the $E_2^{p,q}$ term of a spectral sequence which converges to $L^{p+q}_{B \cap H,G}(-)$. Indeed transitivity of induction implies that $(-) \mid^G_{B \cap H} = (-) \mid^G_H \circ (-) \mid^H_{B \cap H}$ which leads to a spectral sequence of derived functors which we refer to as a spectral sequence of induction.

Since $E_2^{p,q}(-)$ preserves finite dimensionality for all p, q, so does $E_\infty^{p,q}(-)$, it being a subquotient of the former. But then $\bigoplus_{p+q=n} E_\infty^{p,q}(-)$ preserves finite dimensonality for each n, and so $L^n_{B \cap H,G}(-)$ preserves finite dimensionality for all n. Suppose the conjecture holds for solvable groups. Then $B \cap H$ is parabolic and solvable, hence $B \cap H$ is a Borel subgroup B'. But then $B' = B \cap H \subseteq B$ and all Borel subgroups have the same dimension, hence $B' = B$. We have shown that $B \cap H = B$ and so $B \subseteq H$. But then H is parabolic, since it contains a Borel subgroup of G.

Thus we see that in any characteristic, Conjecture A reduces to the case H is solvable. For the remainder of this section we concentrate on some particular solvable subgroups of G. Namely H will be the semidirect product of T with U_J, the unipotent radical of P_J. We regard this as a generalization of the case when J is empty where H is just $T \cdot U = B$.

There is a well-known approach to show that induction $(-)|_B^G$ preserves finite dimensionality which is based on properties of cyclic G-modules generated by maximal vectors, combined with a result [7] from invariant theory. Recall that a maximal vector is a U-fixed weight vector. Accordingly we study cyclic G-modules generated by weight vectors which are fixed by U_J. The necessary result from invariant theory has been extended to this case in [6].

PROPOSITION 2.1. *Let $\hat{V}$ be a rational G-module, and suppose there is a nonzero $v \in \hat{V}_\lambda^{U_J}$ for some $\lambda \in \Lambda(\hat{V})$. Let V be the cyclic G-submodule of $\hat{V}$ generated by v. Then*:

(a) *$\Lambda(V) \subseteq \mathscr{F}_\lambda$, where*

$$\mathscr{F}_\lambda = \{\mu \in \Lambda \mid \exists \mu' \in \lambda + Q_J \text{ such that } \mu' - \mu \in \Lambda_+(\Phi^+ - \Phi_J^+)\}.$$

(b) $\lambda \in W_J(\Lambda_+) = \{w(\mu) \mid \mu \in \Lambda_+, w \in W_J\}$.

(c) *V is generated by a U_J-fixed weight vector with a dominant weight.*

(d) *Assume $\lambda \in \Lambda_+$ without loss of generality by* (c). *Then V is finite dimensional and has a filtration $V = V_n \supseteq V_{n-1} \supseteq \cdots \supseteq V_0 = 0$ such that V_i/V_{i-1} is a cyclic submodule of V/V_{i-1} generated by a maximal vector v_i of weight $\mu_i \in \Lambda_+$ and such that $\mu_i \in \lambda + Q_J^+$. Moreover if $\mu_i \in \mu_j + Q_J^+$ then $j \geq i$.*

(e) *Suppose Φ is an irreducible root system and J is a proper subset of Δ. Then $\lambda = 0$ implies $V \cong k$.*

Proof. Let A_λ be the subspace of $\hat{V}$ spanned by weight vectors whose weights belong to $\mathscr{F}_\lambda$. Define a morphism of varieties $f: G \to \hat{V}$ by $f(g) = g \cdot v$. This is a morphism because $\hat{V}$ is rational, and $\mathrm{Im}(f)$ is the orbit $G \cdot v$ in $\hat{V}$, whose linear span is V. Let $u_1 l u_2$ be an arbitrary element of $U_J^- L_J U_J = U^- P_J$. Then $f(u_1 l u_2) = u_1 l u_2 \cdot v = u_1 l \cdot v$ because U_J fixes v. But $l \cdot v$ is necessarily a sum of weight vectors with weights in $\lambda + Q_J$. Clearly u_1 sends each such weight vector to a sum of weight vectors with weights in $\mathscr{F}_\lambda$, by considering how each $U_{-\gamma}$ acts, where $\gamma \in \Phi^+ - \Phi_J^+$. Thus $u_1 l \cdot v$ is a sum of weight vectors in A_λ, so $f(U_J^- P_J) \subseteq A_\lambda$. But $U_J^- P_J$ is dense in G, whence $f(G) \subseteq A_\lambda$. But if $\mathrm{Im}(f) \subseteq A_\lambda$ so is its linear span V, so $\Lambda(V) \subseteq \mathscr{F}_\lambda$ as claimed in (a).

Define $\mathscr{A} = \{\mu \in \Lambda \mid \langle \mu, \gamma \rangle \geqslant 0$ for all $\gamma \in \Phi^+ - \Phi_J^+\}$ and also define $\mathscr{B} = \{\mu \in \Lambda \mid \langle \mu, \gamma \rangle \geqslant 0$ for all $\gamma \in W_J(\Delta - J)\}$, where $W_J(\Delta - J) = \{w(\gamma) \mid w \in W_J$ and $\gamma \in \Delta - J\}$. Then recall that the elements of W_J all send some positive roots from Φ_J^+ into negative roots, while permuting the remainder of the positive roots $\Phi^+ - \Phi_J^+$. In particular if $\gamma \in \Delta - J$ and $w \in W_J$ then $w(\gamma) \in \Phi^+ - \Phi_J^+$, so if $\mu \in \mathscr{A}$ then automatically $\mu \in \mathscr{B}$. Notice that when J is empty both $\mathscr{A}$ and $\mathscr{B}$ reduce to the dominant weights Λ_+. In fact if J is nonempty we see $\Lambda_+ \subseteq \mathscr{A}$ in any case, but $\mathscr{A}$ is W_J-stable because W_J permutes $\Phi^+ - \Phi_J^+$ and $\langle -, - \rangle$ is W-invariant. Thus we obtain $W_J(\Lambda_+) \subseteq W_J(\mathscr{A}) = \mathscr{A}$, and $\mathscr{A} \subseteq \mathscr{B}$ from above.

Now we claim $\mathscr{B} \subseteq W_J(\Lambda_+)$. Indeed if $\mu \in \mathscr{B}$ then $\langle \mu, \gamma \rangle \geqslant 0$ for all $\gamma \in W_J(\Delta - J)$, so $\langle w(\mu), \gamma \rangle \geqslant 0$ for all $\gamma \in \Delta - J$ and all $w \in W_J$. But this says that every W_J-conjugate of μ is $(\Delta - J)$-dominant. On the other hand exactly one of these conjugates is J-dominant. Indeed write $\mu = \mu_1 + \mu_2$ with $\mu_1 \in E_J$ and μ_2 orthogonal to E_J. Let Λ_J denote the weight lattice for the root system Φ_J in E_J. Note that precisely one W_J-conjugate of μ_1 belongs to $(\Lambda_J)_+$ beause W_J acts simply transitively on the chambers of E_J, while W_J acts trivally on μ_2. Moreover μ is J-dominant in Λ if and only if $\mu_1 \in (\Lambda_J)_+$, so we see exactly one W_J-conjugate of μ belongs to $\Lambda_+^J \cap \Lambda_+^{\Delta + J} = \Lambda_+$ so $\mu \in W_J(\Lambda_+)$ as claimed. Hence $W_J(\Lambda_+) \subseteq \mathscr{A} \subseteq \mathscr{B} \subseteq W_J(\Lambda_+)$ and so they are all equal.

Now by (a), the only weights V contains which are greater than λ are actually greater than λ in the J-relative partial order. Of course $s_\alpha(\lambda) \in \Lambda(V)$ for each simple $\alpha \in \Delta$, so if α is not an element of J then $\langle \lambda, \alpha \rangle \geqslant 0$, beause $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha$. This shows $\lambda \in \Lambda_+^{\Delta - J}$. When J is empty so that $T \cdot U_J = T \cdot U = B$, this is the standard method to show $\lambda \in \Lambda_+$. However, when $J \neq \phi$ then $\Lambda_+^{(\Delta - J)}$ is too large and contains weights which could not possibly correspond to a U_J-fixed vector. By considering $\langle \lambda, \gamma \rangle$ for all $\gamma \in \Phi^+ - \Phi_J^+$ (the roots of U_J), not just for $\gamma \in \Delta - J$ we obtain $s_\gamma(\lambda) = \lambda - \langle \lambda, \gamma \rangle \gamma \in \Lambda(V)$. But γ is not an element of $\Phi_J^+ \Rightarrow \langle \lambda, \gamma \rangle \geqslant 0$ as above. But this shows $\langle \lambda, \gamma \rangle \geqslant 0$ for all $\gamma \in \Phi^+ - \Phi_J^+$ so $\lambda \in \mathscr{B}$ which we know is the same as $W_J(\Lambda_+)$, proving (b).

For (c), merely note that if n represents $w \in W_J$ with $w(\lambda) \in \Lambda_+$, then V is also generated by $n \cdot v \in V_{w(\lambda)}$, since v is in the G-orbit of $n \cdot v$. Also note $n \cdot v$ is U_J-fixed because n normalizes U_J. Thus we may replace v by $n \cdot v$ without loss of generality to assume λ is dominant.

For (d) observe that every vector of $\hat{V}$ lies in a finite dimensional submodule because $\hat{V}$ is rational, so V is finite dimensional. Choose a weight vector v_1 with weight μ_1 maximal in the set $\{\mu \in \Lambda(V) \mid \mu \in \lambda + Q_J^+\}$. If $U_\gamma \subseteq U_J$ and $u \in U_\gamma$, then $u \cdot v_1 = v_1 + \sum_{n>0} c_n v_{\mu_1 + n\gamma}$ where $v_{\mu_1 + n\gamma} \in V_{\mu_1 + n\gamma}$.

If $c_n \neq 0$ then V contains a weight vector of weight $\mu_1 + n\gamma$ which is strictly greater than μ_1, and μ_1 belongs to $\lambda + Q_J^+$; so $\mu_1 + n\gamma \in \lambda + Q^+$. But again by (a) the only weights of V greater than λ are greater in the J-relative partial order so $\mu_1 + n\gamma \in \lambda + Q_J^+$. But $\mu_1 \in \lambda + Q_J^+$ so $n\gamma \in Q_J$, a contradiction because $\gamma \in \Phi^+ - \Phi_J^+$ implies γ is not in the subspace E_J. Thus each $c_n = 0$ and so u fixes v_1. Hence U_J fixes v_1 and by the maximality of $\mu_1, v_1 \in V^{U_\gamma}$ for $\gamma \in \Phi_J^+$ as well so v_1 is fixed by all of U and thus v_1 is a maximal vector of V; $v_1 \in V_{\mu_1}^U$ with $\mu_1 \in \lambda + Q_J^+$.

Let V_1 be the cyclic submodule of V generated by v_1. It may be the case that $v \in V_1$, so V_1 is all of V and the filtration has only one term. Otherwise apply the same process to V/V_1. Since $\Lambda(V/V_1) \subseteq \Lambda(V)$ and $(V/V_1)_\lambda \neq 0$, we may find a nonzero $\bar{v}_2 \in (V/V_1)_{\mu_2}$ with $\mu_2 \in \lambda + Q_J^+$ and maximal in $\Lambda(V/V_1)$ with respect to this property. Moreover μ_2 is not greater than μ_1 in the J-relative partial order, since then $V_{\mu_2} \neq 0$ in contradiction to the maximality of μ_1. Let $\bar{V}_2$ be the cyclic G-module of V/V_1 generated by $\bar{v}_2$, and let V_2 be the inverse image of $\bar{V}_2$ under the quotient map $V \to V/V_1$. Continuing in this manner, we build a filtration $V_1 \subseteq V_2 \subseteq \cdots$ such that V_i/V_{i-1} is a cyclic submodule of V/V_{i-1} generated by a maximal vector v_i of weight $\mu_i \in \lambda + Q_J^+$. Also $\mu_i \in \mu_j + Q_J^+ \Rightarrow j \geqslant i$ by maximality at each stage. Since $\Lambda(V)$ is finite, this filtration terminates eventually. Let V_n be the last term of the filtration. Then $(V/V_n)_\lambda = 0$, or else we could extend the filtration. But then $v \in V_n$ so $V_n = V$ as claimed.

Now (e) follows from (d). Indeed if J is empty, then V is a maximal vector of weight 0 and it is known that such a vector generates k as a G-module. In the more general case, let $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n$ be the filtration of (d). Each μ_i is an element of $\lambda + Q_J^+ = Q_J^+$ when $\lambda = 0$, while also being dominant so $\mu_i \in Q_J^+ \cap \Lambda_+$. But if $J \neq \Delta$ and Φ is an irreducible root system, then we leave it to the reader to check that $Q_J^+ \cap \Lambda_+ = \{0\}$. Thus each $\mu_i = 0$ so in particular $\lambda = 0$ is already maximal in $\{\mu \in \Lambda(V) \mid \mu \in Q_J^+\}$, so v is itself a maximal vector. But again, a G-module generated by a maximal vector of weight 0 is just k. ∎

THEOREM 2.2. *Let G be a reductive algebraic group, let $J \subseteq \Delta$, but $J \neq \Delta$ and let $H = T \cdot U_J$. Then*:

(a) $k[G]^H \cong k\,|_H^G \cong k$.

(b) $(-)\,|_H^G$ *preserves finite dimensionality.*

(c) *If $V \in \mathrm{RAT}(H)$ with $-\Lambda(V) \cap W_J(\Lambda_+) = \phi$, then $V\,|_H^G = 0$.*

Proof. We prove the theorem in the case when G is almost simple so that Φ is an irreducible root system. The reader may easily extend it first to the case when G is semisimple and then to the reductive case by a repeated application of 4.1 of [5].

To prove (a), note that $k\,|_H^G \cong k[G]^H \cong (k[G]^{U_J})^T \cong k[G]_0^{U_J}$. Let f be a nonzero function from this space. Then by part (e) of the last proposition, the cyclic G-submodule generated by f is isomorphic to k. In particular all of U fixes f, so f belongs to $k[G]^B \cong k[G/B]$, the coordinate ring of a projective variety, so f must be a constant function. Thus $k[G]^H \cong k[G]^B \cong k$, as claimed. One could also use the results of [12] to show (a).

For (b) and (c), we induct on $\dim(V)$. If V is one-dimensional, $V \cong -\lambda$ for some $\lambda \in \Lambda$, so we have T-module isomorphisms,

$$-\lambda\,|_H^G \cong (k[G] \otimes -\lambda)_0^{U_J} \cong (k[G]^{U_J} \otimes -\lambda)_0 \cong k[G]_\lambda^{U_J} \otimes -\lambda,$$

as U_J acts trivially on $-\lambda$. If $\lambda = 0$ then $\lambda\,|_H^G$ is finite dimensional by (a), so assume $\lambda \neq 0$. The main result in [6] states that $k[G]^{U_J}$ is a finitely generated k-algebra. Choose generators $\{f_i\}$ for $i = 1, 2, ..., r$ such that $f_i \cdot t = \mu_i(t) f_i$ for some $\mu_i \in \Lambda$. Then $-\mu_i \in W_J(\Lambda+)$ for all i. Indeed let $\hat{V}$ be $k[G]$ under the action $f \to f \cdot g^{-1}$. Then $f_i \in \hat{V}_{-\mu_i}$ and is U_J-fixed, so let V_i be the cyclic submodule generated by f_i and apply proposition 2.2(b) to obtain $-\mu_i \in W_J(\Lambda_+)$ for all i. Moreover, we may assume $\mu_i \neq 0$ since by (a) only the constant functions are H-invariant. Thus $k[G]_\lambda^{U_J}$ is the space spanned by all monomials $\prod_{i=1}^r f_i^{m_i}$ with $\lambda = \sum_{i=1}^r m_i(-\mu_i)$ for some nonnegative integers m_i. Since each $-\mu_i \in W_J(\Lambda_+)$ and $m_i \geq 0$, we see that this equation is satisfied only when $\lambda \in W_J(\Lambda_+)$. This proves (c) if V is one-dimensional. Moreover, (b) follows provided we can show that there are only finitely many solutions to the equation $\lambda = \sum_{i=1}^r m_i(-\mu_i)$ for $m_i \geq 0$ and $\lambda \in W_J(\Lambda_+)$. But notice that for any $\gamma \in \Lambda$ we have $\langle \lambda, \gamma \rangle = \sum_{i=1}^r m_i \langle -\mu_i, \gamma \rangle$. Suppose we can find a γ with $\langle -\mu_i, \gamma \rangle \geq 1$ for each i. Then the latter equation implies that $\langle \lambda, \gamma \rangle \geq m_i \langle -\mu_i, \gamma \rangle \geq m_i$ (because each term in the sum is nonnegative) and so m_i is bounded above by $\langle \lambda, \gamma \rangle$ for each i. Since there eare only finitely many nonnegative integral combinations with the coefficients bounded, there are indeed only finitely many solutions (and $(1 + \langle \lambda, \gamma \rangle)^r$ is a coarse upper bound on the number of solutions). It remains to show such a γ exists.

Let $\gamma = \sum_{\alpha_i \in \Delta - J} \omega_i$ where ω_i is the fundamental dominant weight dual to α_i. Note that $\gamma \neq 0$ because $J \neq \Delta$, while the case J empty gives $\gamma = \rho$. The reader may check that $\langle \mu, \gamma \rangle \geq 0$ for all $\mu \in W_J(\Lambda_+)$, with equality iff $\mu = 0$, so that γ has the desired properties. This establishes (b) if $V \cong -\lambda$.

In general for a finite dimensional V, find an H-stable line in V (possible because H is solvable) spanned by a weight vector V of weight $\lambda \in \Lambda(V)$. So there is a short exact sequence of H-modules with $\Lambda(V) = \Lambda(V/\lambda) \cup \{\lambda\}$:

$$0 \to \lambda \to V \to V/\lambda \to 0.$$

Apply induction to obtain a long exact sequence:

$$0 \to \lambda\,|_H^G \to V\,|_H^G \to (V/\lambda)\,|_H^G \to \cdots.$$

Now by induction on the dimension of V, (b) and (c) hold for λ and V/λ, and the long sequence shows that $V|_H^G$ is also finite dimensional so (b) holds in general. It also shows that (c) holds for finite dimensional modules V, so the general case of (c) follows by a direct limit argument. ∎

In contrast to this theorem, induction from H to P_J rather than to G fails to preserve finite dimensionality. Indeed, if $J \neq \phi$, take $\lambda = 0$ and observe $HL_J = P_J$, so 4.1 of [5] implies

$$k\,|_H^{P_J}\,|_{L_J} \cong k\,|_{H\cap L_J}^{L_J} \cong k\,|_T^{L_J} \cong k[L_J/T]$$

which is the coordinate ring of the affine space L_J/T. Since J is nonempty, this variety is at least one-dimensional so its coordinate ring is an infinite dimensional k-space. The next result takes care of parabolics between P_J and G.

COROLLARY 2.3. *Suppose* $J \subseteq K \subseteq \Delta$ *with* J *nonempty, and let* $H = T \cdot U_J$. *Then* $(-)\,|_H^{P_K}$ *preserves finite dimensionality unless* $J = K$.

Proof. As in the theorem, it suffices to consider the case $V \cong -\lambda$. We have $-\lambda\,|_H^{P_K}\,|_{L_K} \cong -\lambda\,|_H^B\,|_B^{P_K}\,|_{L_K} \cong -\lambda\,|_H^B\,|_{B\cap L_K}\,|^{L_K}$ by 4.1 of [5]. Since $J \subseteq K$, $H(B \cap L_K) = B$ so 4.1 of [5] again applies to give

$$-\lambda\,|_H^{P_K}\,|_{L_K} \cong -\lambda\,|_{H\cap(B\cap L_K)}\,|^{B\cap L_K}\,|^{L_K} \cong -\lambda\,|_{T\cdot U_{K,J}}^{L_K},$$

the last isomorphism by transitivity of induction. As noted above, if $J = K$, this is just $-\lambda\,|_T^{L_J}$ which is infinite dimensional in general. If $J \neq K$ then $H \cap (B \cap L_K) \cong T \cdot U_{K,J}$ is the semidirect product of T with the unipotent radical $U_{K,J}$ of a parabolic subgroup of L_K, viz. $P_J \cap L_K$. So $-\lambda\,|_{T\cdot U_{K,J}}^{L_K}$ is finite dimensional by the theorem applied to L_K. ∎

Observe that when J is empty, Theorem 2.2 reduces to the statement that $(-)\,|_B^G$ preserves finite dimensionality, as mentioned above. We use this to show that if P is parabolic, then $L^n_{P,G}(-)$ preserves finite dimensionality for all n.

First, reduce the problem to the case when $P = B$, a Borel subgroup as follows: let B be a Borel subgroup with $B \subseteq P \subseteq G$. Consider the spectral sequence of induction created by $(-)\,|_B^G = (-)\,|_P^G \circ (-)\,|_B^P$. We have $E_2^{p,q}(V) \cong L^p_{P,G}(L^q_{B,P}(V))$, and it converges to $L^{p+q}_{B,G}(V)$.

But if $V \in \mathrm{RAT}(P)$, by the tensor identity we have

$$L^q_{B,P}(V) \cong L^q_{B,P}(k) \otimes V \cong \begin{cases} V, & q = 0 \\ 0, & q > 0. \end{cases}$$

The last isomorphism follows from Theorem 1 in characteristic 0 and Kempf's theorem in prime characteristics. Thus the sequence collapses to an isomorphism $L^n_{P,G}(V) \cong L^n_{B,G}(V\mid_B)$, for all $n \geqslant 0$ and $V \in \mathrm{RAT}(P)$. So it suffices to show $L^n_{B,G}(-)$ preserves finite dimensionality.

Now consider the case $n=0$ and apply Theorem 2.2 in case J is empty. So induction $(-)\mid^G_B$ preserves finite dimensionality and the result is true for $n=0$. We will now use a form of dimension-shifting to induct on n. Let λ be a character of B and write $\lambda = \sum_{\omega_i \in \Omega} n_i \omega_i$ where $n_i \in Z$. Let $r = \mathrm{Max}(\{-n_i\} \cup \{0\})$, so $r \geqslant 0$ and with equality if and only if λ is dominant. Set $\mu = \lambda + r\rho$ where $\rho = \sum_{\omega_i \in \Omega} \omega_i$. An easy calculation shows that μ is dominant, so there exists an irreducible G-module $S(\mu)$ with highest weight μ, where the μ-weight space is a B-submodule. Apply $(-) \otimes -r\rho$ to obtain a short exact sequence of B-modules:

$$0 \to \lambda \to S(\mu) \otimes -r\rho \to Q \to 0.$$

Now induce this sequence up to G, use the tensor identity and Kempf's theorem (or Borel–Weil Bott if char $k=0$) on terms of the form $L^n_{B,G}(-r\rho)$ to obtain:

(a) an exact sequence

$$0 \to \lambda\mid^G_B \to -r\rho\mid^G_B \otimes S(\mu) \to Q\mid^G_B \to L^1_{B,G}(\lambda) \to 0,$$

(b) isomorphisms for $n > 0$:

$$L^n_{B,G}(Q) \cong L^{n+1}_{B,G}(\lambda).$$

By (a), $L^1_{B,G}(\lambda)$ is finite dimensional because Q is finite dimensional and result is true for $n=0$. So by induction on dim V, the result is true for $n=1$. But then it is true for all n by (b) and induction on n. So $L^n_{P,G}(-)$ preserves finite dimensionality for all n. We mention this approach only because from the point of view of representation theory, it is desirable to have arguments which use as little sheaf cohomology as possible.

3. The Characteristic 0 Case

In this section, G is a semisimple or reductive group defined over an algebraically closed field of characteristic 0. H is a closed connected solvable subgroup of G containing T. In particular H lives inside some Borel subgroup B.

Lemma 3.1. *Let $T \subseteq H \subseteq B$, where H is connected and strictly smaller* *than B. Choose $H' \supseteq H$ with $H'/H \cong U_\beta$ for some $\beta \in \Phi^+$. Then for any $\lambda \in \Lambda$ we have*

$$\chi(\lambda \mid_H^B) = \chi(\lambda \mid_{H'}^B) + \chi((\lambda - \beta) \mid_H^B).$$

Proof. First consider the case $\lambda = 0$, for which we have $k \mid_H^B \cong k[B/H] \cong A_S$, where $S = \Phi(B/H) = \{\gamma \in \Phi^+ \mid U_\gamma \text{ is not contained in } H\}$; and similarly $k \mid_{H'}^B \cong A_{S'}$, where $S' = \Phi(B/H')$. Note that $S = S' \cup \{\beta\}$, so the character computation from Section 1 implies that

$$\chi(k \mid_H^B) = \chi(A_S) = \chi(A_{S'})\chi(k[X_\beta]) = \chi(k \mid_{H'}^B)(1 - e^{-\beta})^{-1}.$$

Multiply both sides by $1 - e^{-\beta}$ to obtain

$$\begin{aligned}\chi(k \mid_H^B) &= \chi(k \mid_{H'}^B) + e^{-\beta}\chi(k \mid_H^B)\\ &= \chi(k \mid_{H'}^B) + \chi(-\beta)\chi(k \mid_H^B)\\ &= \chi(k \mid_{H'}^B) + \chi(k \mid_H^B \otimes -\beta)\\ &= \chi(k \mid_{H'}^B) + \chi(-\beta \mid_H^B),\end{aligned}$$

by the tensor identity. Now multiply both sides by $e^\lambda = \chi(\lambda)$ and use the tensor identity on all three terms. ∎

LEMMA 3.2. *For each $\lambda \in \Lambda$, define a G-module $R_H(\lambda)$ by $R_H(\lambda) = \bigoplus_{j=1}^N R^j$, where*

$$R^j = \bigoplus_{\gamma \in \Lambda(\lambda \mid_H^B)} (L^j_{B,G}(\gamma))^{r_{\lambda,\gamma}}$$

and $N = |\Phi^+|$. Here $L^j_{B,G}(\gamma)^{r_{\lambda,\gamma}}$ denotes a direct sum of $r_{\lambda,\gamma}$ copies of $L^j_{B,G}(\gamma)$, where $r_{\lambda,\gamma}$ is the dimension of $(\lambda \mid_H^B)_\gamma$. Suppose char $k = 0$, *and let μ be a negative dominant weight. Then the multiplicity of the (irreducible) G-module $\mu \mid_B^G$ as a summand of $R_H(\lambda)$ is given by $m_{H,\lambda,\mu} = \sum_{\mu' \in W \cdot \mu} r_{\lambda,\mu'}$.*

Proof. This is immediate from Theorem 1. ∎

LEMMA 3.3. *Let H and H' be as in Lemma 3.1. Then for all $\lambda \in \Lambda$ and $\mu \in -\Lambda_+$ we have*

$$m_{H,\lambda,\mu} = m_{H',\lambda,\mu} + m_{H,\lambda-\beta,\mu}.$$

Proof. $m_{H,\lambda,\mu} = \sum_{\mu' \in W \cdot \mu} \dim(\lambda \mid_H^B)_{\mu'}$, but for any μ' we have $\dim(\lambda \mid_H^B)_{\mu'} = \dim(\lambda \mid_{H'}^B)_{\mu'} + \dim((\lambda - \beta) \mid_H^B)_{\mu'}$ by Lemma 3.1. ∎

LEMMA 3.4. *Let H be a closed, connected subgroup with $T \subseteq H \subseteq B$, and* *suppose $H \neq B$. Then there exists a character λ of T such that there exist infinitely many negative dominant weights μ for which $m_{H,\lambda,\mu} \equiv 1 \pmod 2$.*

Proof. We induct on $d = \dim(B/H)$. For $d = 1$, $B/H \cong U_\alpha$ for some $\alpha \in \Delta$. Then $\lambda \mid_H^B \cong \lambda \otimes k[X_\alpha]$ with formal character $\chi(\lambda \mid_H^B) = e^\lambda(1 + e^{-\alpha} + \cdots)$, so all weight multiplicities are equal to 0 or 1. Note $\Lambda(\lambda \mid_H^B) = \{\lambda - n\alpha \mid n \geqslant 0\}$, which consists of weights which lie on a ray in E with apex λ and in the direction of $-\alpha$. For any such ray there are two possibilities: either it consists entirely of weights which are singular relative to the dot action of W (i.e., each $\lambda - n\alpha$ is on a wall of a chamber for this action of W), or the ray crosses finitely many walls and all the weights lie in the interior of one chamber for n large enough. The reader should have no trouble seeing that λ may be chosen such that the second of these possibilities holds. In particular for $n \gg 0$, $\lambda - n\alpha$ is the only element of its orbit under the dot action of W on E which belongs to $\Lambda(\lambda \mid_H^B)$. But this implies that $r_{\lambda, \lambda - n\alpha} = 1$ for all n and if $n \gg 0$, $r_{\lambda,\mu} = 0$ for all $\mu \in W \cdot (\lambda - n\alpha)$. Let μ_n be the unique negative dominant weight conjugate to $\lambda - n\alpha$ under the dot action of W. Then for $n \gg 0$ we have that $m_{H,\lambda,\mu_n} = 1$ which gives the lemma for $d = 1$.

In general, choose $H' \supseteq H$ with $H'/H \cong U_\beta$ for some root $\beta \in \Phi^+$. Because $\dim(B|H') < \dim(B|H)$ we assume inductively that there is a λ such that there are infinitely many distinct μ's for which $m_{H',\lambda,\mu} \equiv 1 \pmod 2$. But by Lemma 3.3, for each such μ, either $m_{H,\lambda,\mu}$ or $m_{H,\lambda-\beta,\mu}$ is odd. It follows that either infinitely many $m_{H,\lambda,\mu}$ or infinitely many $m_{H,\lambda-\beta,\mu}$ are odd, so either λ or $\lambda - \beta$ works for H. ∎

PROPOSITION 3.5. *Let H be a closed connected subgroup of G with $T \subseteq H \subseteq B$. Suppose the characteristic of k is 0. Then $L^n_{H,G}(-)$ preserves finite dimensionality for all n iff $H = B$.*

Proof. We show the contrapositive, namely that $H \neq B \Rightarrow$ there exists a negative dominant weight for which $L^n_{H,G}(\lambda)$ is infinite dimensional for some n. Observe that $L^n_{H,G}(\lambda) \cong L^n_{B,G}(\lambda \mid_H^B)$ because B/H is affine [3], which forces the spectral sequence of induction created by $(-) \mid_H^G = (-) \mid_B^G \circ (-) \mid_H^B$ to collapse. We now filter $\lambda \mid_H^B$ by it's weight spaces, and compute $L^n_{B,G}(\lambda \mid_H^B)$ by considering the associated spectral sequence of a filtration. This spectral sequence is not a derived functor spectral sequence; in fact it is not even a first quadrant spectral sequence. It is described for cohomology of complexes on page 327 of [11]. See the discussion on page 42 of [14] for a construction of a suitable complex whose nth cohomology group is $L^n_{B,G}(\gamma)$. (Alternatively one could rephrase the proof in terms of the long exact sequences induced by applying $(-) \mid_B^G$ to the various pieces of the filtration.)

In this spectral sequence we have $E_1^{p,q}(\lambda \mid_H^B) \cong L^{p+q}_{B,G}(\gamma)$ for some $\gamma \in \Lambda(\lambda \mid_H^B)$. In other words, $E_1^{*,*}(\lambda \mid_H^B)$ has precisely the same composition factors as $R_H(\lambda)$ of Lemma 3.2. Since each $E_1^{p,q}(\lambda|_H^B)$ is either irreducible or zero, we see each differential $d_1^{p,q}$ is either zero or an isomorphism. $E_2^{*,*}$ consists of all those irreducibles which are not cancelled out when we take cohomology with respect to $d_1^{*,*}$. Each time a nonzero differential $d_1^{p,q}$ occurs, exactly two copies of the same irreducible do not survive to the $E_2^{*,*}$ level. In particular $E_2^{p,q}(\lambda\,|_H^B)$ is either zero or irreducible, and this argument can be iterated to obtain that the multiplicity of $\mu\,|_B^G$ as a composition factor of $E_1^{*,*}(\lambda\,|_H^B)$ has the same parity as it's multiplicity in $E_r^{*,*}(\lambda\,|_H^B)$ for all r. Since the multiplicity $\mu\,|_B^G$ in $E_1^{*,*}(\lambda\,|_H^B)$ is given by $m_{H,\lambda,\mu}$ we obtain the multiplicity of $\mu\,|_B^G$ in $E_\infty^{*,*}(\lambda\,|_H^B)$ is congruent to $m_{H,\lambda,\mu}$ (mod 2).

But $E_\infty^{*,*}(\lambda\,|_H^B)$ has precisely the same composition factors as $\bigoplus_{n=0}^N L_{H,G}^n(\lambda)$. In particular, if $m_{H,\lambda,\mu}$ is odd, then at least one copy of $\mu\,|_B^G$ cannot cancel under the taking of cohomology with respect to any differential so lives forever to become a composition factor of $L_{H,G}^n(\lambda)$ for some n. By Lemma 3.4, when $H \neq B$ there always exists λ's for which there are infinitely many distinct μ's with $m_{H,\lambda,\mu} \equiv 1 \pmod 2$, hence $E_\infty^{*,*}$ has infinitely many distinct composition factors. It follows that $L_{H,G}^n(\lambda)$ is infinite dimensional for some n. ▌

COROLLARY 3.6. *Conjecture A is true in characteristic* 0.

Proof. This follows from Proposition 3.5 and the discussion in Section 2. ▌

4. RELATED RESULTS

We begin by stating an equivalent form of Conjecture A.

Conjecture B. Let H be a closed connected subgroup of a reductive G containing T. Suppose $L_{H,P}^n(-)$ preserves finite dimensionality for all n, where $P \supseteq H$ is any parabolic subgroup of G. Then H is a parabolic subgroup of G.

To see that they are equivalent, note that Conjecture B $\Rightarrow$ Conjecture A, because G is itself parabolic in G. Conversely, if Conjecure A holds, and $H \subseteq P \subseteq G$ is given such that $L_{H,P}^n(-)$ preserves finite dimensionality for all n, consider the spectral sequence of induction created by $(-)\,|_H^G = (-)\,|_P^G \circ (-)\,|_H^P$. We have $E_2^{p,q}(V) = L_{P,G}^p(L_{H,P}^q(V))$ which converges to $L_{H,G}^{p+q}(V)$. Because $L_{H,P}^q(V)$ is finite dimensional for all q, and $L_{P,G}^p(-)$ preserves finite dimensionality for all p, we obtain $E_2^{p,q}(V)$ is finite dimensional for all p, q, hence the same is true for $E_\infty^{p,q}(V)$ and so also for $L_{H,G}^n(V)$ for all n. Thus $L_{H,G}^n(-)$ preserves finite dimensionality for all n so H is parabolic by Conjecture A. Hence Conjecture B holds also.

The reason for restating the conjecture in this form is because of Theorem 4.7 below which shows that in prime characteristics, Conjecture B holds at least in the case of a minimal parabolic of G, so there is some evidence for the validity of Conjecture A in prime characteristics.

We have some more evidence in small rank cases. For example, let G be of type A_1. Then the only connected solvable groups containing T are T itself and a Borel subgroup $B = T \cdot U_\gamma$ where $\gamma \in \Phi$.

But $L^n_{T,G}(-)$ does not preserve finite dimensionality. For example, take $n=0$ and $V=k$ to get $k\,|^G_T \cong k[G]^T \cong k[G/T]$. Since T is a reductive subgroup of G, G/T is an affine variety [3] of dimension >1, so $k[G/T]$ is an infinite dimensional algebra. Thus the conjecture is true trivially for G of type A_1, in any characteristic.

Next let G be reductive, $T \subseteq H \subseteq B$, and let $\lambda \in \Lambda$. Define the Euler characteristic $E_H(\lambda)$ by

$$E_H(\lambda) = \sum_{n=0}^{N} (-1)^n \dim L^n_{H,G}(\lambda),$$

where $N = |\Phi^+|$. Recall that $L^n_{H,G}(\lambda) \cong L^n_{B,G}(\lambda\,|^B_H)$ for all n. Thus $E_H(\lambda) = E_B(\lambda\,|^B_H)$, and the Euler characteristic on G/B (recall the identification with the sheaf cohomology groups) is well known to be independent of char k.

If char k $=0$ and $H \neq B$ we know we can find a λ with $\dim L^n_{H,G}(\lambda) = \infty$ for some n, because Conjecture A is true. Suppose we can show such a λ exists with the further property that $E_H(\lambda)$ is infinite, for example, by finding a λ such that $L^n_{H,G}(\lambda)$ is infinite dimensional for precisely one value of n. Since $E_H(\lambda)$ is independent of char k, it follows that $L^n_{H,G}(-)$ cannot preserve finite dimensionality for all n in prime characteristics either. Then the conjecture would hold also in prime characteristics. This technique is a feasible approach if rank $G=2$. For example, we have:

THEOREM 4.1. *Conjecture* A *is true for* $G = SL_3(k)$ *in all characteristics.*

Proof. The root system for G is type A_2, with simple roots $\Delta = \{\alpha, \beta\}$. As in the characteristic zero case we may assume that H is solvable: $T \subseteq H \subseteq B$. If $\Phi(H)$ is empty then $H = T$, but $k\,|^G_T$ is infinite dimensional in any characteristic, so H does not satisfy the hypothesis of the conjecture. On the other extreme if $\Phi(H) = \Phi^+$ then $H = B$ and there is nothing to prove, so we may assume $1 \leqslant |\Phi(H)| \leqslant 2$. If $|\Phi(H)| = 2$, then not both U_α and U_β are subgroups of H, hence $H \cong T \cdot U_{\{\alpha\}}$ or $T \cdot U_{\{\beta\}}$. By symmetry we may assume $H \cong T \cdot U_{\{\alpha\}}$. By Theorem 2.2, $(-)\,|^G_H$ preserves finite dimensionality in this case. If $\lambda \in -Q_+$ then

$$L^n_{H,G}(\lambda) \cong L^n_{B,G}(\lambda\,|^B_H) \cong L^n_{B,G}(\lambda \otimes k[X_\alpha]).$$

One checks that every weight γ of $\lambda\,|_H^B$ has the property that $L^n_{B,G}(\gamma)=0$ if $n\geqslant 2$. Hence $E_H(\lambda)=\dim(\lambda\,|_H^G)-\dim L^1_{H,G}(\lambda)$ and $\lambda\,|_H^G$ is finite dimensional so if $L^1_{H,G}(\lambda)$ is infinite dimensional in characteristic 0 it is infinite dimensional in characteristic p as well. But there are λ's for which $L^1_{H,G}(\lambda)$ is infinite dimensional in characteristic zero by Proposition 3.5.

Now consider the case $|\Phi(H)|=1$. Note that if $\lambda\in -Q_+$ then it is still true that $L^n_{H,G}(\lambda)=0$ for $n\geqslant 2$, so $E_H(\lambda)=\dim(\lambda\,|_H^G)-\dim L^1_{H,G}(\lambda)$. First consider the case $H=T\cdot U_{\alpha+\beta}$. Then every weight multiplicity of $(\lambda\,|_H^B)\cong k\,|_H^B\otimes\lambda$ is equal to 1, because $k\,|_H^B\cong k[X_\alpha]\otimes k[X_\beta]$ and α and β are linearly independent. Consider $\lambda=\rho=\alpha+\beta$. By inspection we see $\Lambda(\rho\,|_H^B)$ is contained in $\rho+(-\Lambda_+\cup s_\alpha(-\Lambda_+)\cup s_\beta(-\Lambda_+))\cap\rho-Q_+$ where s_α and s_β are the simple reflections in the Weyl group associated to α and β. Moreover if $\mu=-n\rho$, we see that $\mu|_B^G$ occurs once in the "0th degree" part R^0 of $R_H(\rho)$ and twice in the "first degree" part R^1, since $\mu\,|_B^G\cong L^1_{B,G}(s_\alpha\cdot\mu)\cong L^1_{B,G}(s_\beta\cdot\mu)$ by Theorem 1. This shows $m_{H,\rho,\mu}=3$ for every $\mu=-n\rho$ $(n=0,1,2,\ldots)$ and so $L^1_{H,G}(\rho)$ is infinite dimensional. If $\mu\neq -n\rho$, write $\mu=-(r_1\omega_1+r_2\omega_2)$ for some nonnegative r_i, and assume without loss of generality that $r_1<r_2$. (ω_1 is the fundamental dominant weight dual to α, and ω_2 is dual to β.) Then $s_\alpha(\mu)\in\Lambda(\rho\,|_H^B)$ but $s_\beta(\mu)$ is not, so $m_{H,\rho,\mu}=2$ if $r_1\neq r_2$. Moreover, $E_H(\lambda)$ is the Euler characteristic of $E_\infty^{*,*}(\lambda\,|_H^B)$ from Proposition 3.5, which is the same as the Euler characteristic of $E_1^{*,*}(\lambda\,|_H^B)=R_H(\lambda)$ because taking Euler characteristics commutes with taking cohomology. That is to say, $E_H(\lambda)$ is the alternating sum $\sum_{j=0}^N(-1)^j\dim R^j$. Thus for $\lambda=\rho$ we see in fact that $E_H(\rho)=-\sum_{n=0}^\infty\dim(-n\rho\,|_H^G)=-\infty$ since every occurrence of $\mu\,|_B^G$ in R^0 pair with exactly one occurrence in R^1 if $\mu\neq -n\rho$, and pairs with two occurrences in R^1 if $\mu=-n\rho$. Thus ρ has the property that $E_H(\rho)$ is infinite as desired.

It remains to handle the case $H=T\cdot U_\alpha$ or $T\cdot U_\beta$. Using similar methods it is easy to show that $E_H(k)=\dim(k\,|_H^G)-\dim L^1_{H,G}(k)=\sum_{n=0}^\infty\dim(-n\rho\,|_B^G)=\infty$. This completes the proof since for all relevant subgroups H, either the hypothesis of the conjecture is not met, or else we have exhibited a λ with $E_H(\lambda)=\pm\infty$. ∎

Using similar techniques it can be shown that the conjecture is also true for groups of type B_2. We have not made a serious attempt in case G is of type G_2 because of the rather large number of relevant subgroups.

We now look at a different way of using the finiteness property to characterize parabolics. This approach involves just the induction functors and not the higher derived functors, but works with a whole family of parabolics rather than a single one. Let $\Phi(H)=\{\gamma\in\Phi\mid U_\gamma\subseteq H\}$ and $\Delta_H=\Delta\cap\Phi(H)$.

LEMMA 4.2. *Let $T\subseteq H\subseteq B$ and suppose $k\,|_H^{P_J}$ is finite dimensional for some nonempty $J\subseteq\Delta$. Then $\Phi(H)\cap\Phi_J^+\neq\phi$.*

Proof. If $\Phi(H) \cap \Phi_J^+$ is empty, then $\Phi(H) \subseteq \Phi^+ - \Phi_J^+$ or $H \subseteq T \cdot U_J$, where U_J is the unipotent radical of P_J. Thus $k \,|_{T \cdot U_J}^{P_J} \subseteq k \,|_H^{P_J}$ and we have seen (Corollary 2.3) that $k \,|_{T \cdot U_J}^{P_J}$ is infinite dimensional for nonempty J. ∎

COROLLARY 4.3. *Let $T \subseteq H \subseteq B$ and suppose there is a minimal parabolic P_α (for some $\alpha \in \Delta$) such that $k \,|_H^{P_\alpha}$ is finite dimensional. Then $\alpha \in \Delta_H$.*

Proof. Since $\alpha \in \Delta$ the result follows from 4.2 with $J = \{\alpha\}$. ∎

COROLLARY 4.4. *Let H be a closed connected subgroup of G containing T. Then H is parabolic iff there is a Borel subgroup B of G such that $k \,|_{H \cap B}^{P_\alpha}$ is finite dimensional for all $\alpha \in \Delta$.*

Proof. If H is parabolic, there exists a Borel subgroup $B \subseteq H$ and $H \cap B = B$. Thus $k \,|_{H \cap B}^{P_\alpha} = k \,|_B^{P_\alpha} \cong k$ for all $\alpha \in \Delta$. Conversely if there exists such a B with all $k \,|_{H \cap B}^{P_\alpha}$ finite dimensional, then 4.3 gives $\alpha \in \Delta_{H \cap B}$ for all $\alpha \in \Delta$. Thus $U_\alpha \subseteq H \cap B$ for all $\alpha \in \Delta$. But then $H \cap B$ contains the group generated by these U_α, which is the unipotent radical U of B. Thus $B = T \cdot U \subseteq H \cap B \subseteq B$ so $B = H \cap B$ and $H \supseteq B$. Hence H is parabolic. ∎

This characterization says roughly that H is parabolic in G if induction to each minimal parabolic preserves finite dimensionality. But is it really necessary to use all the P_α's? We may cut down on the number of functors involved as the next lemma shows:

LEMMA 4.5. *Suppose $T \subseteq H \subseteq B$ and that $k \,|_H^{P_\alpha}$ is finite dimensional for some α. Then Δ_H contains α and every simple root not adjacent to α on the Dynkin diagram for Φ.*

Proof. If $k \,|_H^{P_\alpha}$ is finite dimensional it follows (see 7.9 of [14]) from Wedderburn's Theorem that it must actually be isomorphic to k. Now $k \,|_H^B \cong k[U/U_1] \cong \bigotimes_{\gamma \in C} k[U_\gamma]$ as k-algebras and T-modules, where $C = \Phi^+ - \Phi(H)$. Clearly k is a B-stable subspace of this B-module, so we may define the rational B-module Q by the short exact sequence:

$$0 \to k \to k \,|_H^B \to Q \to 0. \tag{4.5.1}$$

Induce (4.5.1) up to P_α to obtain a short exact sequence of P_α-modules:

$$0 \to k \to k \,|_H^{P_\alpha} \to Q \,|_B^{P_\alpha} \to 0. \tag{4.5.2}$$

This is exact on the right by the version of Kempf's theorem for P_α (see 4.6 of [14]). But the first map in (4.5.2) is an isomorphism by our assumption, so $Q \,|_B^{P_\alpha} = 0$. Suppose α is not in Δ_H, so $\alpha \in C$. Then the polynomial ring $k[X_\alpha]$ is a tensor factor of $k \,|_H^B$. Thus X_α corresponds to a weight vector of weight $-\alpha$ in $k \,|_H^B$. Clearly, the B-submodule generated by

X_α contains only multiples of X_α and the constant functions, so the coset $\tilde{X}_\alpha = X_\alpha + k$ in Q represents a nonzero maximal vector of weight $-\alpha$ in Q. Thus we have a nonzero B-homomorphism $-\alpha \to Q$, which induces up to an injection $-\alpha \mid_B^{P_\alpha} \to Q \mid_B^{P_\alpha}$ because induction is left exact. But $\alpha \in \Lambda_+^{\{\alpha\}}$ so $-\alpha \mid_B^{P_\alpha} \neq 0$, a contradiction because $Q \mid_B^{P_\alpha} = 0$. Thus $\alpha \in \Delta_H$ and this gives another proof of Corollary 4.3.

However, we can go further, because if β is any simple root not in Δ_H, then the same argument gives a nonzero B-homomorphism $-\beta \to Q$. Now if $\beta \neq \alpha$ and is not adjacent to α on the Dynkin diagram, then $\langle \beta, \alpha \rangle = 0$ so $-\beta \in \Lambda_+^{\{\alpha\}}$; in fact $-\beta$ is a character of P_α. Thus we get $-\beta \mid_B^{P_\alpha} \cong -\beta \subseteq Q \mid_B^{P_\alpha}$, which gives the same contradiction. ∎

COROLLARY 4.6. *Suppose H is a closed connected subgroup of G containing T. Then H contains a Borel subgroup B (so is parabolic) iff there exists an $\alpha \in \Delta$ such that $k \mid_{H \cap B}^{P_\alpha}$ is finite dimensional and such that $k \mid_{H \cap B}^{P_\beta}$ is finite dimensional for every β which is adjacent to α.*

Proof. Apply Lemma 4.5 to $H \cap B$. ∎

In particular if α is an end node for the Dynkin Diagram, and β is the immediately adjacent node, only two modules $k \mid_{H \cap B}^{P_\alpha}$ and $k \mid_{H \cap B}^{P_\beta}$ need be checked. This cuts down the number of P_α involved to two. This is the best one expects, since there are known examples of nonparabolic subgroups with $T \subseteq H$ and yet $(-) \mid_H^G$ preserves finite dimensionality. So it is unlikely that $(-) \mid_{H \cap B}^{P_\alpha}$ preserving finite dimensionality by itself is enough to force H to be parabolic. The above results say we can force H to be parabolic if we vary α. Now we look at the situation where P_α is fixed but we consider $L^n_{H, P_\alpha}(-)$ for all n. That is, we are back in the situation of Conjecture B.

THEOREM 4.7. *Let H be a closed connected subgroup of G containing T. Then H contains a Borel subgroup B (so is parabolic) iff there is a simple root α such that $L^n_{H \cap B, P_\alpha}(k)$ is finite dimensional for all n.*

Proof. Suppose $H \cap B \neq B$, so that $\Delta_H \neq \Delta$. Since P_α is minimal, only $L^0_{H \cap B, P_\alpha}(k)$ and $L^1_{H \cap B, P_\alpha}(k)$ are nonzero. Since $k \mid_{H \cap B}^{P_\alpha}$ is finite dimensional, by 4.5 we know that Δ_H contains α and all $\beta \in \Delta$ not adjacent to α. Thus there is a $\beta \in \Delta_H$ with $\langle \beta, \alpha \rangle < 0$. Note the sequence (4.5.1) fits into a commutative diagram of B-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k & \longrightarrow & k \mid_{H \cap B}^{B} & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow \cong & & \downarrow g & & \\
0 & \longrightarrow & k \mid_{T \cdot U_{\{\beta\}}}^{B} & \longrightarrow & k \mid_{H \cap B}^{B} & \longrightarrow & Q_1 & \longrightarrow & 0
\end{array}$$

In this diagram f is injective, g is surjective, and $\mathrm{Cok}(f) \cong \mathrm{Ker}(g)$ by the snake lemma. We will apply induction to this whole diagram. Observe that the spectral sequence of induction with $E_2^{p,q}(V) = L^p_{B,P_\alpha}(L^q_{H \cap B,B}(V))$ and converging to $L^{p+q}_{H \cap B,P_\alpha}(V)$ actually collapses to an isomorphism $L^n_{H \cap B,P_\alpha}(V) \cong L^n_{B,P_\alpha}(V\,|^B_{H \cap B})$ because $B/H \cap B \cong U/U_2$ is affine [3], where U_2 is the unipotent radical of $B \cap H$. Using this and the fact that if $k\,|^{P_\alpha}_B$ is finite dimensional it must be isomorphic to k, we obtain $Q\,|^{P_\alpha}_B$ and $(\mathrm{Ker}(g))\,|^{P_\alpha}_B$ are both 0, as in 4.5. Moreover the same isomorphism also shows that $L^n_{H \cap B,P_\alpha}(V) = 0$ if $n > 1$, so upon applying $(-)\,|^{P_\alpha}_B$ to the diagram above we obtain first that $k\,|^{P_\alpha}_{T \cdot U_{\{\beta\}}} \cong k$, and the following exact sequence:

$$0 \to Q_1\,|^{P_\alpha}_B \xrightarrow{\delta} L^1_{T \cdot U_{\{\beta\}},P_\alpha}(k) \to L^1_{H \cap B,P_\alpha}(k) \to L^1_{B,P_\alpha}(Q_1) \to 0.$$

(Where we have replaced $L^1_{B,P_\alpha}(k\,|^B_{T \cdot U_{\{\beta\}}})$ by $L^1_{T \cdot U_{\{\beta\}},P_\alpha}(k)$, which is in fact also isomorphic to $L^1_{B,P_\alpha}(\mathrm{Ker}(g))$ because $L^1_{B,P_\alpha}(k) = 0$.) Next observe that the weights of $\mathrm{Ker}(g)$ all have the form $-n\beta$ for some $n > 0$, as $k\,|^B_{T \cdot U_{\{\beta\}}} \cong k[X_\beta]$. Since β is adjacent to α, we have $\langle -n\beta, \alpha \rangle > 0$, so every weight of $\mathrm{Ker}(g)$ is in $\Lambda^{\{\alpha\}}_+$. In particular $L^0_{B,P_\alpha}(-n\beta) = 0 = L^2_{B,P_\alpha}(-n\beta)$, so $L^1_{B,P_\alpha}(\mathrm{Ker}(g))$ is filtered by the modules $L^1_{B,P_\alpha}(-n\beta)$ as n ranges through 1, 2, 3, ... Each of these is nonzero in any characteristic by Serre duality, so $L^1_{B,P_\alpha}(\mathrm{Ker}(g))$ is infinite dimensional. However, $L^1_{H \cap B,P_\alpha}(k)$ is finite dimensional by hypothesis, so reference to the exact sequence above shows that $Q_1\,|^{P_\alpha}_B$ is infinite dimensional. In particular, since every weight multiplicity of Q_1 is finite, we obtain $\Lambda(Q_1) \cap -\Lambda^{\{\alpha\}}_+$ is infinite by 4.11 of [14]. However, we can say more than that, namely that $\Lambda(Q_1)$ contains μ_n for $n \gg 0$, where μ_n is some weight with μ_n less than or equal to $s_\alpha \cdot (-n\beta)$ in the partial order for P_α; that is, $\mu_n + r\alpha = s_\alpha \cdot (-n\beta)$ for some $r \geqslant 0$. Indeed we have $\Lambda(L^1_{T \cdot U_{\{\beta\}},P_\alpha}(k)) = \bigcup_{n=1}^\infty \Lambda(s_\alpha \cdot (-n\beta)\,|^{P_\alpha}_B)$ by Serre duality, which contains $s_\alpha \cdot (-n\beta)$ for all n. However from the above exact sequence we see that by assumption $\mathrm{Cok}(\delta)$ is finite dimensional, hence $\Lambda(\mathrm{Im}(\delta))$ contains $s_\alpha \cdot (-n\beta)$ for $n \gg 0$; i.e., $\Lambda(Q_1\,|^{P_\alpha}_B)$ contains $s_\alpha \cdot (-n\beta)$ for $n \gg 0$. But if $\mu \in \Lambda(Q_1)$, $\mu\,|^{P_\alpha}_B$ contains only weights which are greater than or equal to μ in this partial order, hence the claim follows by filtering Q_1 by its weight spaces. Note that $\mu_n \in \Lambda(Q_1) \cap E_{\{\alpha,\beta\}}$.

Let $J = \{\alpha, \beta\}$ and observe that $\Lambda(k\,|^B_H) = \Lambda(k[B/H]) \subseteq -Q_+$, hence $\Lambda(k\,|^B_H) \cap E_J \subseteq -(Q_J)_+$. It follows that the only monomials X^n_γ in $k[B/H]$ with weights in E_J are those with $\gamma \in \Phi^+_J$. Next observe that α is adjacent to β implies that $\langle -\beta, \alpha \rangle \in \{1, 2, 3\}$. Let M be the subspace of Q spanned by the monomials in Q of the form $X_{\beta + n\alpha}$ for $n = 0$ 1, ..., $\langle -\beta, \alpha \rangle$ (that is, monomials corresponding to the α-string through β). Note that $X_\gamma \in Q$ if and only if γ is not an element of $\Phi(H)$. Because β is not an element of Δ_H, $X_\beta \in Q$ and so $M \neq 0$. Note also that a monomial of the form $X^s_\alpha X^t_\gamma$ belongs to the algebra $k[B/H]$ if and only if X_α and X_γ both belong to $k[B/H]$, as the latter is a polynomial ring. It follows that any weight vector of Q of weight $-(\beta + r\alpha)$ for $r \leqslant \langle -\beta, \alpha \rangle$ must be a multiple of $X_{\beta + r\alpha}$. (Because the only other possibilities are multiples of monomials of the form $X_{\beta + s\alpha} X_\alpha^q$ where $s + q = r$, but these do not belong to $k[B/H]$, as X_α is not an element of $k[B/H]$.) We now claim that the subspace M is a B-submodule of Q. Indeed if $\gamma \in \Phi^+$, express $\varepsilon = \gamma - (\beta + r\alpha)$ in terms of the simple roots. If ε has both negative and positive coefficients, or all nonnegative coefficients, then ε is either not a root or a member of Q_+, and in either case not a weight of $Q \cong k[B/H]/k$. Hence U_γ fixes $X_{\beta + r\alpha}$. On the other hand, if $\varepsilon \in -Q_+$, being less than $-(\beta + r\alpha)$, it must be of the form $-(\beta + r_1\alpha)$ with $r_1 < r$, or of the form $-r_1\alpha$, with $r_1 < r$. In the first case, either $-(\beta + r_1\alpha)$ is not a weight of $k[B/H]$ and U_γ fixes $X_{\beta + r\alpha}$ or else U_γ sends $X_{\beta + r\alpha}$ to itself plus multiples of the monomials belonging to M and in the second case U_γ sends $X_{\beta + r\alpha}$ to itself plus multiples of monomials of the form X_α^n for some n. However, X_α is not a monomial in Q because $\alpha \in \Delta_H$, so U_γ fixes $X_{\beta + r\alpha}$ after all. In every case we have shown $U_\gamma(M) \subseteq M$ for all $\gamma > 0$, so $B(M) \subseteq M$.

We now claim that the condition that $\Lambda(Q_1)$ contains μ_n for $n \gg 0$ (together with the assumption that β is not an element of Δ_H) implies that $r\alpha + \beta$ is not an element of $\Phi(H)$ for all $r = 0, 1, ..., \langle -\beta, \alpha \rangle$. The proof of this breaks up into cases depending on $\langle -\beta, \alpha \rangle$. We prove the claim in the case when $\langle -\beta, \alpha \rangle = 3$, and also finish the proof of the theorem in that case by getting a contradiction to the assumption that β is not in Δ_H. The other two cases are handled similarly and are left for the reader.

So assume that $\langle -\beta, \alpha \rangle = 3$, so that the root system Φ_J is of type G_2 and α is the short root. Suppose that $\beta + 3\alpha \in \Phi(H)$, so that the monomial $X_{\beta + 3\alpha}$ does not belong to Q. Then as $\Lambda(k[B/H]) \subseteq \Lambda_+(\Phi^+ - \Phi(H))$ and both α and $\beta + 3\alpha$ belong to $\Phi(H)$, we see $\Lambda(k[B/H]) \subseteq \Lambda_+(\{\beta, \beta + \alpha, \beta + 2\alpha, 2\beta + 3\alpha\})$. On the other hand, $s_\alpha \cdot (-n\beta) = -n(3\alpha + \beta) + \alpha$, and for $n > 1$, one easily sees that neither $s_\alpha \cdot (-n\beta)$ nor μ_n is a member of that cone (draw a weight diagram!). As $\Lambda(Q_1) \subseteq \Lambda(k[B/H])$, this is a contradiction because $\mu_n \in \Lambda(Q_1)$ for $n \gg 0$. Thus $\beta + 3\alpha$ is not an element of $\Phi(H)$, and hence neither is $\beta + r\alpha$ for any $r \leqslant 3$. (Indeed if $U_{\beta + r\alpha} \subseteq H$ and $U_\alpha \subseteq H$, it follows that $U_{\beta + 3\alpha} \subseteq H$.) Thus M is a four-dimensional B submodule of Q and the subspace M_1 spanned by monomials $X_{\beta + r\alpha}$ for $r < 3$ is a B stable subspace of M of dimension 3 and the subspace M_2 spanned by monomials with $r < 2$ is a B-submodule of M_1 of dimension of 2. Thus we have short exact sequences of B-modules:

$$0 \to M_1 \to M \to -(\beta + 3\alpha) \to 0,$$

$$0 \to M_2 \to M_1 \to -(\beta + 2\alpha) \to 0,$$

$$0 \to -\beta \to M_2 \to -(\beta + \alpha) \to 0.$$

Now apply $(-)\,|_B^{P_\alpha}$ to each of these sequences and note that $M_i \subseteq Q$ implies $M_i\,|_B^{P_\alpha}=0$ for each i because we know that $Q\,|_B^{P_\alpha}=0$. From the last of these sequences we get $L^1_{B,P_\alpha}(-\beta)\cong L^1_{B,P_\alpha}(M_2)$ because $\langle -(\beta+\alpha),\alpha\rangle = +1$ implies $L^n_{B,P_\alpha}(-(\beta+\alpha))=0$ for all n (Theorem 1(a)). But Serre duality implies that $L^1_{B,P_\alpha}(-\beta)$ is isomorphic to $-(\beta+2\alpha)\,|_B^{P_\alpha}$ which is three dimensional, so $L^1_{B,P_\alpha}(M_2)$ is three dimensional. From the second sequence we get the exact sequence:

$$0\to -(\beta+2\alpha)\,|_B^{P_\alpha}\to L^1_{B,P_\alpha}(M_2)\to L^1_{B,P_\alpha}(M_1)\to 0.$$

Since the first two terms are three dimensional, we obtain $L^1_{B,P_\alpha}(M_1)=0$. However, from the first sequence we get $0\neq -(\beta+3\alpha)\,|_B^{P_\alpha}\subseteq L^1_{B,P_\alpha}(M_1)$, a contradiction. Thus in this case we see that β must also be in Δ_H if β is adjacent to α. Since we have already shown that $\beta\in\Delta_H$ if β is not adjacent to α or if $\beta=\alpha$, we obtain that $\Delta_H=\Delta$ and so $H\cap B=B$ and H is parabolic. ▮

As remarked above this shows that Conjecture B holds in the case of a minimal parabolic P_α. We conclude by mentioning another well-known property of induction from a parabolic. If $P\subseteq G$ is parabolic and M is an irreducible P-module, then $M\,|_P^G$ (if nonzero) has an irreducible socle. It turns out that this property also characterizes parabolic subgroups. This is discussed in [13].

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