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Support varieties for algebraic groups

By *Daniel K. Nakano* at Athens, *Brian J. Parshall* at Charlottesville, and
David C. Vella at Saratoga Springs

1. Introduction

1.1. Over the last twenty years, geometric techniques have played an important role in studying the representation theory of groups and Lie algebras. For finite groups and restricted Lie algebras over fields of positive characteristic, the cohomological spectrum is a central geometric object in the study of the modular representation theory. Carlson [Ca1], [Ca2] first defined affine algebraic varieties associated to modules over group algebras (called support varieties); these are subvarieties of the cohomological spectrum. The structure of these varieties was later related to "rank varieties" for elementary abelian subgroups [Q], [AS] via a stratification theorem. For restricted enveloping algebras, or equivalently infinitesimal k-groups of height at most one, Friedlander and Parshall [FP1], [FP2] first investigated the theory of support varieties. More recently, Bendel, Friedlander and Suslin [SFB1], [SFB2] have extended this work to more general infinitesimal group schemes. The methods and results in the infinitesimal case often differ considerably from the corresponding theory in the finite group case. Despite much progress in using these varieties for conceptual as well as constructive purposes, there exist few explicit computations of support varieties for specific modules, with the exception of small rank cases [LN1] and induced modules for GL_n (type A) [Jan4].

The present paper provides new results of both a theoretical and an explicit computational nature relative to the determination of support varieties for the infinitesimal subgroups G_r ($r \geqq 1$) of a reductive group G. These results involve the induced modules $H^0(\lambda) = \mathrm{ind}_B^G \lambda$ and $Z_r(\lambda) = \mathrm{ind}_{B_r}^{G_r} \lambda$ (see (1.3) below for notation). In particular, when $r = 1$ we gain an understanding of the relationship between the support varieties of these modules. This leads to the computation of the support variety $V_{G_1}(H^0(\lambda))$ of $H^0(\lambda)$ in all types when the underlying field has good characteristic relative to G; see Theorem (6.2.1) for a precise statement of results. This theorem completely answers a question raised by Jantzen in 1987 [Jan4], (2.7)(1) (now commonly known as the "Jantzen conjecture" on support varieties). Even when the characteristic is not good, our methods provide some new infor-

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mation. Although we obtain the sharpest results in the $r = 1$ case, many results (described more fully below) have been presented for arbitrary $r \geqq 1$. We hope our approach will lead to further progress towards explicit computations in the $r > 1$ case.

The determination of the support variety $V_{G_1}(H^0(\lambda))$ is a fundamental computation which has applications to other areas of modular representation theory. For non-restricted representations of classical Lie algebras, our results in conjunction with results of Gordon and Premet [GP], Thm. 4.2 determine the support variety for the direct sum of simple modules in a given block $\mathscr{B}_\lambda$. Given a G-module, Carlson, Lin, and Nakano [CLN] have defined a finite map of varieties between the support variety for the finite Chevalley group $V_{G(\mathbb{F}_p)}(M)$ and $V_{G_1}(M)$ modulo the action of $G(\mathbb{F}_p)$. This map along with the determination $V_{G_1}(H^0(\lambda))$ has been used to compute the dimension of $V_{G(\mathbb{F}_p)}(H^0(\lambda))$ when $G = \mathrm{GL}_n$ and $n \leqq 5$. Moreover, the results in this paper are used to compute the dimension of the support varieties for all simple modules of Chevalley groups $G(\mathbb{F}_p)$ of Lie rank two (see [CLN], Thm. 4.3). Further applications are presented (in Section 6). In particular, Theorem (6.3.1), shows that for p-good, the p-restricted nullcone $\mathcal{N}_1(G)$ is irreducible. The dimension of this important variety is also computed. These results (along with Theorem (6.2.1)) provide answers to several old questions raised by Friedlander and Parshall [FP1], (3.2), (3.3).

1.2. The paper is organized as follows. Section 2 provides some basic results which will be used throughout the paper. The complexity $c_{G_r}(M)$ of a G_r-module M is the dimension of its support variety $V_{G_r}(M)$. Section 3 develops the analytic methods for obtaining lower bounds on $c_{G_r}(M)$. To this end, we make considerable use of the notion of the generic dimension $\dim_t M$, as developed in [PW]. To be technically precise, $\dim_t M$ is only defined when M lies in the subcategory G_rT-mod of G_r-mod having a compatible action of a maximal torus T of G. Letting $\hat{Z}_r(\lambda) = \mathrm{ind}_{B_rT}^{G_rT}\,\lambda$ and $h_r(t) = \dim_t \hat{Z}_r(0)$, the rational function $\dim_t M/h_r(t)$ has poles located at roots of unity. Further, the order of any such pole is bounded above by $c_{G_r}(M)$. In cases when the character of M can be given, this approach provides an explicit lower bound for $c_{G_r}(M)$. For example, when $M = H^0(\lambda)$, we obtain in Corollary 3.4.3 such a bound using the Weyl character formula. We record here the influence to our approach in Section 3 of the work of V. Ostrik [Ost] in the context of quantum enveloping algebras at an ℓth root of unity, for $\ell > h$ (the Coxeter number).

Our approach makes heavy use of the theory of "relative" support varieties to provide a bridge linking the theory for the $H^0(\lambda)$ with that for the $Z_r(\mu)$. For finite groups, these relative support varieties have already been used to study the complete vanishing of cohomology (see, e.g., [BCRi]). Thus, these methods play an important role in Theorem (4.4.1)(b) which shows that

$$(1.2.1) \qquad V_{G_r}(H^0(\lambda)) \subseteqq \bigcup_{\mu \in \Gamma_r(\lambda)} G \cdot V_{G_r}(Z_r(\mu)),$$

where $\Gamma_r(\lambda)$ denotes the weights corresponding to the G_r-block containing $Z_r(\lambda)$. Also, Theorem (4.6.1) provides necessary and sufficient conditions for (1.2.1) to be an equality of varieties. The end of Section 4 presents an alternate proof of the computation, given in [Jan4], of $V_{G_1}(H^0(\lambda))$ for $G = \mathrm{GL}_n$. In our argument, the inclusion (1.2.1) substitutes for the "weak BGG resolution" method in [Jan4].

Section 5 investigates the behavior of induction on varieties. Using a key relationship on relative support varieties established in Theorem (5.4.1), Theorem (5.6.1) proves in the $r = 1$ case, for any dominant weight λ and any w in the Weyl group W, the containment

$$(1.2.2) \qquad V_{G_1}(H^0(\lambda)) \subseteqq G \cdot V_{G_1}(Z_1(w \cdot \lambda)).$$

Using these two theorems, which comprise the technical heart of our paper, together with some results from Sections 3 and 4, Section 6 answers affirmatively the conjecture raised by Janzten mentioned above.[1] In addition, this section contains further applications of our approach.

This paper provides new information concerning the support varieties $V_{G_r}(Z_r(\lambda))$. In some special cases, e.g., when λ is regular and $r = 1$, these varieties can be calculated; see Theorem (4.1.1). Finally, Theorem (7.3.1) provides a basic link with $H^0(\lambda)$, namely, that in good characteristics we have

$$(1.2.3) \qquad V_{G_1}(H^0(\lambda)) = G \cdot V_{B_1}(Z_1(w \cdot \lambda))$$

for all $\lambda \in X(T)_+$ and all $w \in W$. Since $Z_1(w \cdot \lambda)$ is naturally a G_1B-module, $V_{B_1}(Z_1(\lambda))$ carries an action of the Borel subgroup B. For $w = 1$, we have $\operatorname{ind}_{G_1B}^{G} Z_1(\lambda) \cong H^0(\lambda)$, so the identity (1.2.3) can be regarded as an "induction formula" for certain support varieties. As we note in the example given in (7.1), the varieties $V_{B_1}(Z_1(w \cdot \lambda))$ vary with w. While a determination of the $V_{B_1}(Z_1(w \cdot \lambda))$ does not yet exist, (1.2.3) does tie these varieties together with that of $H^0(\lambda)$.

1.3. Notation. Throughout this paper, let k be an algebraically closed field of positive characterstic p. Let Φ be a finite root system for a Euclidean space $\mathbb{E}$. The inner product on $\mathbb{E}$ will be denoted by $(\ ,\ )$. For $\alpha \in \Phi$, let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ be the corresponding coroot. Fix a set $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ of simple roots, and let Φ^+ be the corresponding set of positive roots. The Weyl group $W \subset O(\mathbb{E})$ is the group generated by the reflections $s_\alpha\colon \mathbb{E} \to \mathbb{E}$, $\alpha \in \Phi$, given by $s_\alpha(x) = x - 2(x, \alpha^\vee)\alpha$. Let $\mathscr{T}_p$ be the group generated by translations $t_{p\alpha}\colon \mathbb{E} \to \mathbb{E}$, $\alpha \in \Phi$, defined by $x \mapsto x + p\alpha$. The affine Weyl group W_p is $W \ltimes \mathscr{T}_p$.

Unless otherwise stated, G will denote a reductive algebraic group defined and split over the prime field $\mathbb{F}_p$. We will always assume that the derived group G' is simply connected. Also, assume that G has root system Φ with respect to maximal split torus T. Let $B \supset T$ be the Borel subgroup defined by $-\Phi^+$. The positive Borel subgroup containing T will be denoted B^+. Moreover, let $X(T) = X(B)$ be the group of integral characters on T or, equivalently, B. Given $\lambda \in X(T)$, we will let λ also denote the one-dimensional B-module defined by regarding λ as a character on B. Then the set of dominant integral weights is defined by

$$X(T)_+ = \{\lambda \in X(T) \mid 0 \leqq (\lambda, \alpha_i^\vee), 1 \leqq i \leqq \ell\}.$$

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[1] [Ost] also claims to provide an answer to this question when p is larger than the Coxeter number h of G (though he works entirely in the context of quantum groups at a root of unity); however, we have been unable to completely check his details (see for example the footnote to Section 3). In any event, our methods provide an alternative approach based on the study of the support varieties of the modules $Z_1(\lambda)$.

(If G is not semisimple, then $\lambda \in X(T)_+$ is not determined by its "values" at the coroots $\alpha^\vee$.) Furthermore, for $r \geqq 1$, the set of p^r-restricted weights is

$$X_r(T) = \{\lambda \in X(T) \mid 0 \leqq (\lambda, \alpha_i^\vee) < p^r, 1 \leqq i \leqq \ell\}.$$

The group W_p acts on $X(T)$ via the "dot action" given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, $w \in W_p$, $\lambda \in X(T)$. Here ρ is the half sum of the positive roots. We partially order $X(T)$ by setting $\lambda \geqq \mu$ if and only if $\lambda - \mu \in \sum_{\alpha \in \Pi} \mathbb{N}\alpha$. Let h be the Coxeter number of G. Thus, if G' is simple, $h = (\rho, \alpha_0^\vee) + 1$ where α_0 is the maximal short root in Φ; otherwise, h is the maximal of the Coxeter numbers for the simple factors of G'. Let

$$C = \{\lambda \in \mathbb{E} \mid 0 < (\lambda + \rho, \alpha_0^\vee) < p\}$$

denote the bottom p-alcove in $\mathbb{E}$. Thus, W_p is generated by the reflections in the walls of C. We set $C_{\mathbb{Z}} = C \cap X(T)$.

The prime p is *good* for Φ (or G) provided that for any (integrally) closed subsystem Φ' of Φ, $\mathbb{Z}\Phi/\mathbb{Z}\Phi'$ has no p-torsion. Equivalently, p is good provided: $p > 2$ when G has a component of type B, C or D; $p > 3$ when G has a component of type G_2, F_4, E_6 or E_7; and $p > 5$ when G has a component of type E_8.

Let $F: G \to G$ be the Frobenius morphism on G induced by its $\mathbb{F}_p$-structure. For $r \geqq 1$, put $G_r = \ker(F^r)$. If H is an F-stable subgroup of G, write similarly $H_r = \ker(F^r|_H)$—e.g., $B_r = \ker(F^r|_B)$. In general, let $G_r H = F^{-r}(H)$, the pull-back of H through F^r—e.g., $G_r T = F^{-r}(T)$. The group scheme G_r is a finite k-group, i.e., an affine algebraic group scheme over k with finite dimensional coordinate algebra $k[G_r]$. Also, it has height $\leqq r$. In what follows, all affine k-groups K will, by definition, be assumed to be algebraic, i.e., the coordinate algebra $k[K]$ is assumed to be finitely generated over k.

For an (affine) k-group H, H-mod denotes the category of rational H-modules. For G above, the dominant weights $\lambda \in X(T)_+$ index the simple modules $L(\lambda)$ by their highest weight. If ind_B^G: B-mod $\to$ G-mod is the induction functor, let $H^0(\lambda) = \mathrm{ind}_B^G \lambda$ for $\lambda \in X(T)$. If $\lambda \notin X(T)_+$, then $H^0(\lambda) = 0$, while if $\lambda \in X(T)_+$ then $H^0(\lambda)$ has socle $L(\lambda)$. For $j \geqq 0$, let $H^j(-) = R^j \mathrm{ind}_B^G(-)$ be the jth right derived functor of ind_B^G.

For $\lambda \in X(T)$ set $Z_r(\lambda) = \mathrm{ind}_{B_r}^{G_r} \lambda$. Then $Z_r(\lambda)$ has irreducible socle, denoted $L_r(\lambda)$. Of course, if $\mu \in X(T)$, $Z_r(\lambda) \cong Z_r(\lambda + p^r\mu)$ and $L_r(\lambda) \cong L_r(\lambda + p^r\mu)$. Thus, the set $X_r(T)$ indexes the distinct simple modules $L_r(\lambda)$ and the distinct $Z_r(\lambda)$. For $\lambda \in X_r(T)$, $L(\lambda)|_{G_r} \cong L_r(\lambda)$.

Any $\lambda \in X(T)$, defines a one-dimensional $B_r T$-module, and $\hat{Z}_r(\lambda) = \mathrm{ind}_{B_r T}^{G_r T} \lambda$ will be the corresponding induced $G_r T$-module. Then $\hat{Z}_r(\lambda)|_{G_r} \cong Z_r(\lambda)$. If

$$\hat{P}_r(\lambda) = \mathrm{ind}_{B_r^+ T}^{G_r T}(\lambda - 2(p^r - 1)\rho),$$

then $\hat{P}_r(\lambda)|_{B_r T}$ (resp., $\hat{P}_r(\lambda)|_{B_r}$) is the projective cover in $B_r T$-mod (resp., B_r-mod) of the one-dimensional module λ. As T-modules, $\hat{Z}_r(\lambda)$ and $\hat{P}_r(\lambda)$ have identical characters given by

$$(1.3.1) \qquad \operatorname{ch} \hat{Z}_r(\lambda) = \operatorname{ch} \hat{P}_r(\lambda) = e^{\lambda} \prod_{\alpha \in \Phi^+} \frac{1 - e^{-p^r \alpha}}{1 - e^{-\alpha}}.$$

For $I \subseteqq \Pi$, let L_I be the reductive subgroup of G generated by T and the root groups U_α, $\pm\alpha \in I$. Let Φ_I be the root system of T in L_I, i.e., $\Phi_I = \mathbb{Z}I \cap \Phi$. The subgroup L_I is a Levi factor for a parabolic subgroup $P_I \supseteqq B$. Let P_I^+ be the opposite parabolic subgroup. The unipotent radical of P_I (resp., P_I^+) will be denoted U_I (resp., U_I^+). For $r \geqq 1$ and $\lambda \in X(T)$, let $Z_r^I(\lambda) = \operatorname{ind}_{(B_I)_r}^{(L_I)_r} \lambda$ and let $Z_r^I(\lambda)^e$ be the $(P_I)_r$-module obtained from $Z_r^I(\lambda)$ by making $(P_I)_r$ act through the quotient map $(P_I)_r \to (P_I)_r/(U_I)_r \cong (L_I)_r$. Then $\operatorname{ind}_{(P_I)_r}^{G_r} Z_r^I(\lambda)^e \cong Z_r(\lambda)$.

In this paper, we work almost exclusively with algebraic varieties in the classical sense. In particular, any commutative, finitely generated k-algebra A defines an affine variety X which, as a set, can be identified with the set of all algebra homomorphisms $f\colon A \to k$. If A_{red} denotes the quotient of A by its nilpotent radical, then X has coordinate algebra $k[X] = A_{\mathrm{red}}$. If $\mathscr{I}$ is an ideal in A, let $V(\mathscr{I})$ be the closed subvariety of X defined by $\mathscr{I}$, consisting of all $f \in X$ such that $f(\mathscr{I}) = 0$.

Often the algebra A will carry an action of a group G. Thus, G acts on the variety X defined by A. Given an ideal $\mathscr{I}$ in A, we let $\ell_G(\mathscr{I})$ denote the largest ideal contained in $\mathscr{I}$ which is G-stable. Clearly,

$$(1.3.2) \qquad \overline{G \cdot V(\mathscr{I})} = V\big(\ell_G(\mathscr{I})\big).$$

In particular, suppose G is a reductive group acting rationally on A and $Y = V(\mathscr{I})$ is a P-stable, closed subvariety of X for some parabolic subgroup $P = P_I$. Let Z be the closed subvariety of $G/P \times X$ consisting of all (gP, x) such that $x \in g \cdot Y$. Since G/P is complete, the projection $G \cdot Y$ of Z to X is closed. Thus, $G \cdot Y = V\big(\ell_G(\mathscr{I})\big)$.

For example, let $I \subseteqq \Pi$ and let $\mathfrak{u}_I = \operatorname{Lie} U_I \subseteqq \mathfrak{g}$. Regard $\mathfrak{g}$ as a G-variety by means of the adjoint action. Then $G \cdot \mathfrak{u}_I$ is a closed subvariety of $\mathfrak{g}$. Also, at least when p is good for G, we have

$$(1.3.3) \qquad \dim G \cdot \mathfrak{u}_I = 2 \dim \mathfrak{u}_I = |\Phi| - |\Phi_I|.$$

See [Car], (5.2.3). In addition, for $I, J \subseteqq \Pi$, we have

$$(1.3.4) \qquad w(\Phi_J) \cap \Phi_I \neq \emptyset \quad \text{for all } w \in W \Leftrightarrow x_I \notin G \cdot \mathfrak{u}_J,$$

where $x_I = \sum_{\alpha \in I} x_\alpha$, with $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in I$. The proof is a simple application of the Bruhat decomposition; see [Jan4], (2.5)(1) for details.

2. Cohomology and support varieties

In this section, we introduce some preliminary results which will be used through the paper.

2.1. Rates of growth. Suppose that $\{s_n\}_{n \geqq 0}$ is a sequence of complex numbers. The *rate of growth* $r(s_n)$ of this sequence is the smallest non-negative integer d for which there exists a positive real number C such that

$$|s_n| \leqq C \cdot n^{d-1}, \quad \forall n \geqq 1.$$

If no such d exists, set $r(s_n) = \infty$.

Suppose s_n is a polynomial in n of degree $d - 1$ (at least for large n). In this case, there exist positive integers C, C' such that

$$C' n^{d-1} \leqq |s_n| \leqq C n^{d-1}, \quad \forall n \text{ sufficiently large.}$$

In particular, $r(s_n) = d$. If $S_n = |s_0| + \cdots + |s_n|$, then it also follows that $r(S_n) = d + 1$.

In practice, $\{s_n\}$ arises as the sequence of coefficients in a power series expansion $\sum_n s_n t^n$ of a rational function $p(t)$. In general, such coefficients s_n are not polynomial functions in n, though the following lemma describes a situation in which $\{s_n\}$ can be broken down into "polynomial subsequences".

(2.1.1) Lemma. *Let* $p(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{C}[[t]]$ *be a rational function.*

(a) *If* $p(t) = \dfrac{f(t)}{(1-t)^d}$ *for some positive integer* d *and* $f(t) \in \mathbb{C}[t]$ *with* $f(1) \neq 0$, *then* s_n *is a polynomial in* n *of degree* $d - 1$. *Hence,* $r(s_n) = d$.

(b) *Assume the poles of* $p(t)$ *are roots of unity. If* $e^{i\theta}$ *is a pole of order* γ, *then*

$$\gamma \leqq r(s_n).$$

(c) *Assume* $p(t) = \dfrac{f(t)}{(1-t^a)^b}$ *for positive integers* a, b *and* $f(t) \in \mathbb{C}[t]$. *Fix* i, $0 \leqq i < a$. *For* j *sufficiently large,* s_{i+aj} *is a polynomial in* j *(say of degree* $d_i - 1$*).*

(d) *In* (c), *let* $d = \max d_i$. *Then* $d = r(s_n)$. *If* $S_n = |s_0| + \cdots + |s_n|$, *then* $r(S_n) = d + 1$.

Proof. (a) Expanding $f(t)$ as a polynomial in $1 - t$ with non-zero constant term, we reduce immediately to the case $p(t) = \dfrac{1}{(1-t)^d} = \sum_{n=0}^{\infty} \binom{n+d-1}{d-1} t^n$. Now (a) follows.

(b) Suppose that $d = r(s_n)$. For a pole $e^{i\theta}$, consider $z = re^{i\theta}$, $0 < r < 1$,

$$|p(z)| \leqq \sum_{n=0}^{\infty} C r^n n^{d-1} \leqq \frac{C'}{(1-r)^d}$$

for some $C, C' > 0$. Thus, the order of the pole of $p(t)$ at $e^{i\theta}$ is at most d.

(c) It is enough to consider the special case in which $f(t) = t^e$ for some non-negative integer e. If $e = i + \ell_0 a$, $0 \leqq i < a$, then $p(t) = \sum_{n=0}^{\infty} \binom{n+b-1}{b-1} t^{i+(n+\ell_0)a}$. It follows that, for $j \geqq \ell_0$, s_{i+aj} equals the polynomial $\binom{j-\ell_0+b-1}{b-1}$ in j.

Finally, (d) follows directly from the remarks before the statement of the lemma. □

Finally, if $\{V_n\}_n$ is a sequence of finite dimensional vector spaces over k, define its rate of growth $r(V_n)$ to be $r(\dim V_n)$.

2.2. Support varieties. Let K be a finite k-group. Set

$$H(K,k) = \begin{cases} H^{2\bullet}(K,k) & \text{if } \operatorname{char} k \neq 2, \\ H^{\bullet}(K,k) & \text{if } \operatorname{char} k = 2. \end{cases}$$

The cohomology algebra $H(K,k)$ is a commutative, finitely generated k-algebra [FS]; thus, we will work with the variety $V_K = \operatorname{Spec} H(K,k)_{\mathrm{red}}$.

Given finite dimensional $M, M' \in K$-mod, define the *relative support variety* $V_K(M, M')$ as follows. First, the functor $- \otimes M$ defines an algebra homomorphism $\Phi_M \colon H(K,k) \to \operatorname{Ext}_K^{\bullet}(M,M)$. Then

$$\Psi\colon H(K,k) \otimes \operatorname{Ext}_K^{\bullet}(M,M') \to \operatorname{Ext}_K^{\bullet}(M,M'), \quad a \otimes b \mapsto \Phi_M(a) \sqcup b,$$

where $\sqcup$ is Yoneda composition, defines an action of $H(K,k)$ on $\operatorname{Ext}_K^{\bullet}(M,M')$. Alternatively, we have $\Psi(a \otimes b) = \pm b \sqcup \Phi_{M'}(a)$, see [Ben], §5.7. Let $J_K(M,M')$ be the annihilator ideal in $H(K,k)$ for its action on $\operatorname{Ext}_K^{\bullet}(M,M')$ and set $V_K(M,M')$ equal to the closed subvariety of V_K defined by $J_K(M,M')$. In particular, the *support variety* $V_K(M)$ is defined by setting $V_K(M) = V_K(M,M)$. In this paper, support varieties $V_K(M)$ or relative support varieties $V_K(M,M')$ are only defined for finite dimensional modules M, M'. The finite dimensionality assumption is always imposed in these cases, though not always explicitly mentioned.

The algebraic variety V_K has coordinate algebra $H(K,k)/J$, where J is the radical of $H(K,k)$. Since J is necessarily a graded ideal, V_K is a homogeneous variety. Any $V_K(M,M')$ is a homogeneous closed subvariety of V_K, and

$$V_K(M,M') \subseteqq V_K(M) \cap V_K(M'). \tag{2.2.1}$$

Let $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ be a minimal projective resolution of a finite dimensional $M \in K$-mod, and define the *complexity* $c_K(M)$ of M to be $r(P_n)$. The following theorem explains the relationship between $c_K(M)$ and $V_K(M)$. A proof is given in [Ben], §5.7.

(2.2.2) Theorem. *Let K be a finite k-group, and let $\{L_i \mid i = 1, 2, \ldots, m\}$ be the complete set of non-isomorphic simple objects in* K-mod. *For finite dimensional $M \in K$*-mod *the following numbers are equal*:

(a) $c_K(M)$;

(b) $\dim V_K(M)$;

(c) $r(\mathrm{Ext}_K^n(M,M))$;

(d) $r\left(\mathrm{Ext}_K^n\left(M, \bigoplus_{i=1}^m L_i\right)\right)$;

(e) $r\left(\mathrm{Ext}_K^n\left(\bigoplus_{i=1}^m L_i, M\right)\right)$.

We have listed below some fundamental properties of complexity and relative support varieties for finite k-groups K [SFB1], [SFB2], [Ben]. In each case, the modules are assumed to be finite dimensional.

(2.2.3) If $M \in K$-mod then $V_K(M)$ is a closed, conical subvariety of V_K.

(2.2.4) If $M_1, M_2 \in K$-mod then $V_K(M_1 \oplus M_2) = V_K(M_1) \cup V_K(M_2)$.

(2.2.5) If $M_1, M_2 \in K$-mod then $V_K(M_1 \otimes M_2) = V_K(M_1) \cap V_K(M_2)$ (set-theoretic intersection).

(2.2.6) Assume that K is a closed normal subgroup of an algebraic k-group L. There is a natural rational action of L on V_K. If $M, N \in L$-mod, the subvariety $V_K(M,N)$ of V_K is stable under the action of L.

(2.2.7) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in K-mod. If $\mathfrak{S}_3$ is the symmetric group on three letters and $\sigma \in \mathfrak{S}_3$, then $V_K(M_{\sigma(1)}) \subseteq V_K(M_{\sigma(2)}) \cup V_K(M_{\sigma(3)})$.

(2.2.8) For any finite-dimensional K-module M, we have $V_K(M) \subseteq \bigcup V_K(L_i)$, where the union runs over the set of composition factors L_i of M.

(2.2.9) If $\{L_1, L_2, \ldots, L_m\}$ is a complete set of non-isomorphic simple modules in K-mod then $V_K(M) = V_K\left(M, \bigoplus_{i=1}^m L_i\right) = V_K\left(\bigoplus_{i=1}^m L_i, M\right)$.

Let H be a closed subgroup of a finite k-group K. Assume K has height $\leqq r$ (e.g., $K = G_r$). Then the image of the restriction map $H(K,k)_{\mathrm{red}} \to H(H,k)_{\mathrm{red}}$ contains all p^rth powers x^{p^r} of elements $x \in H(H,k)_{\mathrm{red}}$ [SFB2], (5.4). In particular, the induced morphism $V_H \to V_K$ maps V_H homeomorphically onto its image as a closed subvariety of V_K. Thus, we often "identify" V_H with its image in V_K. With this identification, we have the following naturality result [SFB2], §7.

(2.2.10) Assume that H is a closed subgroup of a finite k-group K. Identify V_H as a closed subvariety of V_K. Then $V_H \cap V_K(M) = V_H(M)$.

Now let H be an affine algebraic group defined over $\mathbb{F}_p$. Let $\mathcal{N}(H)$ be the closed subvariety of nilpotent elements in the Lie algebra $\mathfrak{h}$ of H. Let $x \mapsto x^{[p]}$ be the $[p]$-operator for the restricted Lie algebra structure on $\mathfrak{h}$. We will often work with the closed subvariety $\mathcal{N}_1(H) = \{x \in \mathfrak{h} \mid x^{[p]} = 0\}$ of $\mathcal{N}(H)$. By [SFB1], (1.6), (5.11), V_{H_1} is homeomorphic to

$\mathcal{N}_1(H)$, and we often identify these two spaces. (When $H = G$ and $p > h$, this homeomorphism is in fact an isomorphism [AJ].) Given $M \in H_1$-mod, any $x \in \mathcal{N}_1(H)$ defines an operator X on M satisfying $X^p = 0$. We then say that M is *x-projective* if M is a projective module for the algebra $k[X] \subseteqq \mathrm{End}(M)$. We have the following result [FP1]:

(2.2.11) Let $M \in H_1$-mod. Under the identification of V_{H_1} with $\mathcal{N}_1(H)$, $V_{H_1}(M)$ identifies with the set $\{x \in \mathcal{N}_1(H) \mid M \text{ is not } x\text{-projective}\} \cup \{0\}$.

When G is reductive and the prime p is good, $\mathcal{N}(G)$ is G-isomorphic to the irreducible variety $\mathcal{U}(G)$ of unipotent elements of G; e.g., see [SS], p. 229, [BR], (9.3.2). In this case, many of the nice properties, such as existence of regular, subregular, etc. elements, of $\mathcal{U}(G)$ transfer to $\mathcal{N}(G)$. This fact will be used in the sequel without mention.

Finally, consider the k-groups G_rT, $r \geqq 1$. Given a finite dimensional G_rT-module M, let $P_\bullet \to M$ be a minimal projective resolution of M in G_rT-mod, define its complexity $c_{G_rT}(M)$ to be the rate of growth of the sequence $\{P_n\}$. Restricting to G_r, it follows easily from [Jan2], II, (11.3)(3) that $P_\bullet \to M$ is also a minimal projective resolution in G_r-mod. This gives the following result for G_rT-mod; a similar argument applies for B_rT-mod. (See also [DNP], (7.2).)

(2.2.12) For any finite dimensional $M \in G_rT$-mod, we have $c_{G_rT}(M) = c_{G_r}(M)$. Similarly, for a finite dimensional $M \in B_rT$-mod, we have $c_{B_rT}(M) = c_{B_r}(M)$.

The reader should observe that the version of (2.2.2), interpreting complexity in terms of the rate of growth of Ext-groups, fails in general for the categories G_rT-mod and B_rT-mod, e.g., $c_{G_rT}(M) \neq r(\mathrm{Ext}^n_{G_rT}(M, M))$ for general finite dimensional $M \in G_rT$-mod.

2.3. We conclude this section with the following result concerning how support varieties behave relative to certain induction functors. We will return to this topic in Section 5.

(2.3.1) Proposition. *Assume the hypotheses of* (2.2.10). *For a finite dimensional* $M \in H$-mod, *we have*

(a) $V_K(\mathrm{ind}_H^K M) \subseteqq V_H(M)$;

(b) $V_K(\mathrm{ind}_H^K M) = V_H(\mathrm{ind}_H^K M)$.

In particular, for any positive integer r, we have $V_{G_r}(Z_r(\lambda)) = V_{B_r}(Z_r(\lambda))$ *for all* $\lambda \in X(T)$. *Also, let* $N \in K$-mod *be finite dimensional. Then the following statement holds for relative support varieties:*

(c) $V_K(N, \mathrm{ind}_H^K M) = V_H(N, M)$.

Proof. By (2.2.10), (a) $\Rightarrow$ (b). By (2.2.1), (c) $\Rightarrow$ (a), taking $N = \mathrm{ind}_H^K M$. Now (c) follows easily from the remarks following (2.2.9), using the fact that since K/H is affine, $\mathrm{Ext}^\bullet_K(N, \mathrm{ind}_H^K M) \cong \mathrm{Ext}^\bullet_H(N, M)$. We leave further details to the reader. □

Of course, $V_{B_r}(M) = V_{U_r}(M)$ for any finite dimensional $M \in B_r$-mod. In particular, it follows from (2.3.1) that $V_{G_r}(Z_r(\lambda)) = V_{U_r}(Z_r(\lambda))$ for all $\lambda \in X(T)$. This variation on (2.3.1) will often be used below in Section 5.

3. Complexity and the graded dimension

Throughout this section G will denote a fixed simple, simply connected algebraic group. The assumption that G is simple is merely one of convenience, and the application of results later to a general reductive group is routine.

3.1. In what follows, we often identify the category of finite dimensional rational T-modules with the category of finite dimensional vector spaces graded by $X(T)$. For an object V in this category, we wish to associate to V a natural $\mathbb{Z}$-grading $V = \bigoplus V_i$ and consider the associated Laurent polynomial $\dim_t V = \sum \dim V_i t^i \in \mathbb{Z}[t, t^{-1}]$. In our presentation, we follow closely the procedure described in detail in [PW], §1.[2] Let $C = ((\alpha_i, \alpha_j^\vee))$ be the Cartan matrix of G. Let $\alpha_0 \in \Phi^+$ be the maximal short root in Φ, and, for any root α, set $d_\alpha = (\alpha, \alpha)/(\alpha_0, \alpha_0) \in \{1, 2, 3\}$. For $1 \leqq i \leqq \ell$, let $d_i = d_{\alpha_i}$, so that the matrix $C \cdot \operatorname{diag}(d_1, \ldots, d_\ell)$ is symmetric. Given $\lambda = \sum_{i=1}^{\ell} n_i \alpha_i \in X(T)$ (so that each $n_i \in \mathbb{Q}$), its *weighted height* is defined to be $\operatorname{wht}(\lambda) = \sum d_i n_i$. A direct calculation shows that $\operatorname{wht}(\lambda) = 2(\lambda, \rho)/(\alpha_0, \alpha_0) = \frac{1}{2} \sum_{\alpha \in \Phi^+} d_\alpha(\lambda, \alpha^\vee)$, so that $2\operatorname{wht}(\lambda) \in \mathbb{Z}$. Now given a finite dimensional $X(T)$-graded vector space V (i.e., a rational T-module) put

$$\text{(3.1.1)} \qquad \dim_t V = \sum_{\lambda \in X(T)} \dim V_\lambda \, t^{-2\operatorname{wht}(\lambda)} \in \mathbb{Z}[t, t^{-1}].$$

In [PW], $\dim_t V$ is called the generic dimension of V (and accordingly denoted $\dim_{\text{gen}} V$ there). We record several useful properties of this concept. First, if $\lambda \in X(T)_+$, then

$$\text{(3.1.2)} \qquad \dim_t H^0(\lambda) = \prod_{\alpha \in \Phi^+} \frac{t^{d_\alpha(\lambda+\rho, \alpha^\vee)} - t^{-d_\alpha(\lambda+\rho, \alpha^\vee)}}{t^{d_\alpha(\rho, \alpha^\vee)} - t^{-d_\alpha(\rho, \alpha^\vee)}}.$$

See [PW], (1.3). Observe that $d_\alpha(\rho, \alpha^\vee) = \operatorname{wht}(\alpha)$. Second, if V, W are $X(T)$-graded finite dimensional vector spaces, then $V \otimes W$ and V^* have natural $X(T)$-gradings and

$$\text{(3.1.3)} \qquad \dim_t(V \otimes W) = \dim_t V \cdot \dim_t W,$$

$$\text{(3.1.4)} \qquad \dim_{t^{-1}} V = \dim_t V^*.$$

Third, define

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[2] Our method for grading rational T-modules differs from the so-called *principal grading* discussed in [K], (10.10) which assigns a $\mathbb{Z}$-grading to any highest weight module $L(\Lambda)$ for a Kac-Moody Lie algebra $\mathfrak{g}(A)$. This grading depends on the highest weight Λ, though a definition could be given for weight modules having all their weights lying in a cone of weights. Such gradings could be adopted in our current framework, and, in fact, [Ost] appears to use some version of the principal grading in the similar context of quantum groups, though the gradings there are not explicitly defined. The use of $\dim_t$ avoids any difficulty of working with a cone of weights and presents an explicit $\mathbb{Z}$-grading for any rational T-module.

$$h_r(t) = \prod_{\alpha \in \Phi^+} \frac{1 - t^{2p^r \operatorname{wht}(\alpha)}}{1 - t^{2\operatorname{wht}(\alpha)}} \in \mathbb{Z}[t]. \tag{3.1.5}$$

Then for any $\lambda \in X(T)$, (1.3.1) implies

$$\dim_t \hat{P}_r(\lambda) = \dim_t \hat{Z}_r(\lambda) = t^{-2\operatorname{wht}(\lambda)} h_r(t). \tag{3.1.6}$$

For any finite dimensional T-module M, define $e(M)$ to be the smallest integer such that $t^{e(M)} \dim_t M \in \mathbb{Z}[t]$. Observe that for every weight λ of M, we have $e(M) - 2\operatorname{wht}(\lambda) \geqq 0$.

3.2. Consider the algebra $A = H(B_r, k)$ for some fixed integer $r \geqq 1$. The argument in [CPSK], (2.1), (2.2) with the normalized bar resolution shows that as a T-module A has all weights in $p^r\mathbb{Z}\Phi^+$. Thus, for all integers $n \geqq 0$,

$$\dim_t H^n(B_r, k) \in \mathbb{Z}[t^{2p^r}, t^{-2p^r}]. \tag{3.2.1}$$

For a finite dimensional $M \in B_rT$-mod, let $P_\bullet \to M \to 0$ be a minimal projective resolution in B_rT-mod. Restricting to B_r, $P_\bullet \to M \to 0$ still provides a minimal projective resolution in B_r-mod. Because $\operatorname{Hom}_{B_r}(P_n, \mu) \cong \operatorname{Ext}^n_{B_r}(M, \mu)$, we have

$$P_n \cong \bigoplus_{\mu \in X_r(T)} \operatorname{Ext}^n_{B_r}(M, \mu)^* \otimes \hat{P}_r(\mu) \tag{3.2.2}$$

as T-modules. Of course, $\operatorname{Ext}^n_{B_r}(M, \mu)$ is a direct sum of one-dimensional T-modules of the form $p^r\nu$.

Let $\lambda \in X_r(T)$ and $\nu \in X(T)$ be such that $\hat{P}_r(\lambda) \otimes p^r\nu$ is a B_rT-direct summand of P_n. Then $\tau_n \overset{\text{def}}{=} \lambda + p^r\nu$ appears in the head of P_n. Since $P_\bullet$ is a minimal projective resolution, it follows that τ_n is a weight in the radical of P_{n-1}. Thus, $\tau_{n-1} \overset{\text{def}}{=} \tau_n + \beta$ appears in the head of P_{n-1}, where β is some non-trivial sum of positive roots. In particular,

$$-2\operatorname{wht}(\tau_n) \geqq -2\operatorname{wht}(\tau_{n-1}) + 2.$$

Repeating this argument n times, we conclude that $-2\operatorname{wht}(\lambda + p^r\nu) \geqq -2\operatorname{wht}(\tau) + 2n$ for some weight $\tau = \tau_0$ in M. Since $e(M) - 2\operatorname{wht}(\tau) \geqq 0$ for all weights τ in M, we finally conclude that

$$-2\operatorname{wht}(\lambda + p^r\nu) \geqq 2n - e(M). \tag{3.2.3}$$

3.3. For a positive integer d let $\Psi_d(t) \in \mathbb{Z}[t]$ be the dth cyclotomic polynomial over $\mathbb{Q}$. The polynomials $\Psi_d(t)$ are irreducible and $t^n - 1 = \prod_{d|n} \Psi_d(t)$. The following result (influenced by [Ost]) shows the complexity $c_{B_r}(M) = \dim V_{B_r}(M)$ of a module M is related to the orders of the poles of the rational function $\dim_t M / h_r(t)$.

(3.3.1) Theorem. *For a finite dimensional* $M \in B_rT$-mod, *write*

$$q(t) = \frac{\dim_t M}{h_r(t)} = \frac{f(t)}{(1 - t^{p^r})^{\gamma} g(t)}$$

where $\Psi_{p^r}(t) \nmid f(t)g(t)$. *Then* $\gamma \leqq c_{B_r}(M)$.

Proof. We can assume $\gamma \geqq 0$, so that γ is the order of the pole of any primitive p^rth root of unity in $q(t)$. Let $P_{\bullet} \to M$ be a minimal projective resolution in B_rT-mod, and write

$$\dim_t P_n / h_r(t) = \sum_m a(m,n) t^m,$$

where each $a(m,n)$ is a non-negative integer, and, using (3.2.2),

$$\sum_m a(m,n) = \dim \operatorname{Ext}^n_{B_r}(M,N) \leqq Cn^{c_{B_r}(M)-1}. \tag{3.3.2}$$

Here $N = \bigoplus_{\mu \in X_r(T)} \mu$ and $C > 0$. If $a(m,n) \neq 0$, then for some $\lambda \in X_r(T)$ and $v \in X(T)$, we have that $\hat{P}_r(\lambda + p^r v)$ appears as a B_rT-direct summand of P_n with $m = -2\,\mathrm{wht}(\lambda + p^r v)$. Thus, (3.2.3) implies that if $a(m,n) \neq 0$ then $m \geqq 2n - e(M)$. In particular, if $a(m,n) \neq 0$, then $n \leqq m + e(M)$. Thus, $q(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\dim_t P_n}{h_r(t)}$. Express $q(t) = \sum_{i=0}^{\infty} s_i t^i$ as a power series, so for any $i > 0$, (3.3.2) gives that

$$\begin{aligned} S_i = |s_0| + \cdots + |s_i| &\leqq \sum_{m=0}^{i} \sum_{n=0}^{\infty} a(m,n) \leqq \sum_{m=0}^{i} \sum_{n=0}^{i+e(M)} a(m,n) \\ &\leqq \sum_{n=0}^{i+e(M)} Cn^{c_{B_r}(M)-1} \leqq C' i^{c_{B_r}(M)} \end{aligned} \tag{3.3.3}$$

for some $C' > 0$.

By (3.1.5), the poles of $q(t)$ are roots of unity. So if a is the least common multiple of the orders of these poles, $q(t) = \frac{m(t)}{(1 - t^a)^d}$ for some positive integer d and some $m(t) \in \mathbb{C}[t]$. By (2.1.1)(c), $q(t) = q_0(t) + \cdots + q_{a-1}(t)$, with $q_i(t) = \sum_j a_{ij} t^{i+ja}$ so that a_{ij} is a polynomial in j of degree $d_i - 1$. Let $d = \max d_i$. By (2.1.1)(d), $r(S_i) = d + 1$ and $r(s_i) = d$. Thus, by (3.3.3), $r(s_i) \leqq c_{B_r}(M)$. Now (3.3.1) follows from (2.1.1)(b). □

3.4. For the root system Φ, define

$$d(\Phi^+, p^r) = |\{\alpha \in \Phi^+ \mid d_\alpha(\rho, \alpha^\vee) = \mathrm{wht}(\alpha) \in p^r \mathbb{Z}\}|.$$

If $p^r \geqq h$, the tables in [B], pp. 250–275 can be used to readily check that $d(\Phi^+, p^r) = 0$. Set $d(\Phi, p^r) = 2d(\Phi^+, p^r)$.

By using (3.3.1), we obtain a relationship between the complexity $c_{B_r}(M)$ of a B_rT-module M with the multiplicity of the cyclotomic polynomial $\Psi_{p^r}(t)$ as a divisor of $\dim_t M$.

(3.4.1) Theorem. *Let* $M \in B_rT$-mod *be finite dimensional.*

(a) *Let* s *be a positive integer such that* $\Psi_{p^r}(t)^s \nmid \dim_t M$ (*resp.*, $\Psi_{2^{r+1}}(t)^s \nmid \dim_t M$) *if* $p > 2$ (*resp.*, $p = 2$). *Then*

$$c_{B_r}(M) \geqq |\Phi^+| - d(\Phi^+, p^r) - s + 1.$$

(b) *Now suppose that* $r = 1$. *If* $M \in G$-mod *and* s *is as in* (a), *then*

$$c_{G_1}(M) = 2c_{B_1}(M) \geqq |\Phi| - d(\Phi, p) - 2s + 2.$$

Proof. Assume $p \neq 2$. By (3.1.5), $\Psi_{p^r}(t)$ is a divisor of $h_r(t)$ with multiplicity $|\Phi^+| - d(\Phi^+, p^r)$. Thus, (3.3.1) implies that $\Psi_{p^r}(t)$ is a divisor of $\dim_t M$ with multiplicity at least $|\Phi^+| - d(\Phi^+, p^r) - c_{B_r}(M)$. Hence, if

$$\Psi_{p^r}(t)^s \nmid \dim_t M, \quad s \geqq |\Phi^+| - c_{B_r}(M) - d(\Phi^+, p^r) + 1.$$

This proves (a), while (b) follows from this and [LN2], (3.4): if $M \in G$-mod then $\frac{1}{2}c_{G_1}(M) = c_{B_1}(M)$. Finally, if $p = 2$, the above argument for (a) works using $\Psi_{2^{r+1}}(t)$. ☐

For $\lambda \in X(T)$ and $r \geqq 1$, let $\Phi_{\lambda, p^r} = \{\alpha \in \Phi \mid d_\alpha(\lambda + \rho, \alpha^\vee) \in p^r\mathbb{Z}\}$. Set

$$\Phi^+_{\lambda, p^r} = \Phi_{\lambda, p^r} \cap \Phi^+,$$

so that $|\Phi_{\lambda, p^r}| = 2|\Phi^+_{\lambda, p^r}|$. For any $w \in W_p$, if $\bar{w}$ denotes the image of w under the quotient map $W_p \to W_p/\mathscr{T}_p = W$, then

$$\Phi_{w \cdot \lambda, p^r} = \bar{w}(\Phi_{\lambda, p^r}). \tag{3.4.2}$$

(3.4.3) Corollary. *Let* $\lambda \in X(T)_+$. *Then*:

(a) $c_{B_r}(H^0(\lambda)) \geqq |\Phi^+| - |\Phi^+_{\lambda, p^r}|$.

(b) *For* $r = 1$, $c_{G_1}(H^0(\lambda)) \geqq |\Phi| - |\Phi_{\lambda, p}|$.

Proof. We give the proof for $p \neq 2$, leaving the modification for $p = 2$ to the reader. By (3.1.2), $|\Phi^+_{\lambda, p^r}| - d(\Phi^+, p^r)$ is the multiplicity of $\Psi_{p^r}(t)$ as a divisor of $\dim_t H^0(\lambda)$. Thus, (3.4.1)(a) implies that

$$\begin{aligned} c_{B_r}(H^0(\lambda)) &\geqq |\Phi^+| - d(\Phi^+, p^r) - (|\Phi^+_{\lambda, p^r}| - d(\Phi^+, p^r)) \\ &= |\Phi^+| - |\Phi^+_{\lambda, p^r}|. \end{aligned}$$

This proves (a), while (b) follows from (a) and (3.4.1)(b). ☐

We will show in (6.2.2) that the inequality in (3.4.3)(b) is actually an equality.

3.5. The following result relates the p-divisibility of the module with the dimension of the support variety.

(3.5.1) Theorem. *We have*:

(a) *If* $M \in B_rT$-mod *is finite dimensional and* $a = |\Phi^+| - d(\Phi^+, p^r) - c_{B_r}(M) \geqq 0$, *then* $p^a \mid \dim M$.

(b) *Suppose* $r = 1$. *If* $M \in G$-mod *is finite dimensional and* $b = |\Phi| - d(\Phi, p) - c_{G_1}(M)$, *then* $p^{b/2} \mid \dim M$.

Proof. First, assume $p > 2$. By (3.4.1)(a), $\Psi_{p^r}(t)^a \mid \dim_t M$. Since

$$\Psi_{p^r}(t) = 1 + t^q + \cdots + t^{q(p-1)} \quad \text{for } q = p^{r-1}, \ p^a = \Psi_{p^r}(1)^a$$

divides $\dim M$, proving (a). Next, $\frac{b}{2} = \frac{1}{2}\big(|\Phi| - c_{G_1}(M)\big) = |\Phi^+| - c_{B_1}(M) = a$, so (a) $\Rightarrow$ (b). If $p = 2$, replace $\Psi_{p^r}(t)$ by $\Psi_{2^{r+1}}(t)$. $\square$

4. Comparing supports of Verma and induced modules

4.1. A weight $\lambda \in X(T)$ is *p-regular* if $(\lambda + \rho, \alpha^\vee) \notin p\mathbb{Z}$ for all $\alpha \in \Phi$. The condition $p \geqq h$ is equivalent to the existence of p-regular weights. In particular, by the definition of h, the zero weight 0 is p-regular if and only if $0 \in X(T)_+ \cap C_{\mathbb{Z}}$ if and only if $p \geqq h$. If $p > h$, then V_{G_1} is isomorphic to $\mathcal{N}_1(G)$ which coincides with the variety $\mathcal{N}(G)$ of nilpotent elements in $\mathfrak{g}$. If $\mathcal{N}_1(G) = \mathcal{N}(G)$, it is true that $p \geqq h$.[3)]

For $\alpha \in \Pi$, let $0 \neq x_\alpha \in \mathfrak{g}_\alpha$. For $I \subseteq \Pi$, let $x_I = \sum_{\alpha \in I} x_\alpha$. Assume that $p \geqq h$, so that $x_\Pi \in \mathcal{N}_1(G)$. In this section, we will make several uses of [Jan1], (4.14)(1), which says the following: Assume that $p \geqq h$ and assume that $\lambda \in X(T)_+$ is not p-regular. Then $L(\lambda)$ is x_Π-projective. Thus, for all p, if $\lambda \in X(T)_+$ is not p-regular, then $x_\Pi \notin V_{G_1}\big(L(\lambda)\big)$; in particular, $V_{G_1}\big(L(\lambda)\big) \subsetneqq \mathcal{N}(G)$. In (6.4.1), we will give another proof of this result.

The next result computes $V_{G_1}\big(Z_1(\lambda)\big)$ when λ is p-regular.

(4.1.1) Theorem. *Let* $p \geqq h$. *Then* $\lambda \in X(T)$ *is p-regular if and only if*

$$V_{G_1}\big(Z_1(\lambda)\big) = \mathcal{N}_1(U).$$

Proof. By [AJ], (2.9), $\operatorname{Ext}^n_{B_1}(k, w \cdot 0) \cong \operatorname{Ext}^{n-l(w)}_{B_1}(k, k)$, so

$$\begin{aligned} r\big(\operatorname{Ext}^n_{B_1}(k, w \cdot 0)\big) &= r\big(\operatorname{Ext}^{n-l(w)}_{B_1}(k, k)\big) \\ &= r\big(\operatorname{Ext}^n_{B_1}(k, k)\big) \\ &= \dim \mathcal{N}_1(U). \end{aligned}$$

[3)] This fact can be checked in a case-by-case basis. It is obvious in type A. Otherwise, one shows that, if $p < h$, then $x^{[p]} \neq 0$ for x a regular nilpotent element (e.g., x is the sum of the simple root vectors). In classical types B, C, D and type G_2, this check is easy. A more laborious verification is required in the remaining exceptional cases. On the other hand, the claim follows immediately from (6.3.1) below.

Furthermore, the universal mapping property of induction and (2.2.2) imply that

$$\begin{aligned} c_{G_1}(Z_1(w\cdot 0)) &= r\left(\operatorname{Ext}^n_{G_1}\left(\bigoplus_{\lambda\in X_1(T)} L_1(\lambda), Z_1(w\cdot 0)\right)\right) \\ &\geqq r\big(\operatorname{Ext}^n_{G_1}(k, Z_1(w\cdot 0))\big) \\ &= r\big(\operatorname{Ext}^n_{B_1}(k, w\cdot 0)\big) \\ &= \dim \mathcal{N}_1(U). \end{aligned}$$

However, by (2.3.1), $V_{G_1}(Z_1(w\cdot 0)) = V_{B_1}(Z_1(w\cdot 0)) \subseteqq \mathcal{N}_1(U)$. It follows that

$$c_{G_1}(Z_1(w\cdot 0)) = \dim V_{G_1}(Z_1(w\cdot 0)) \leqq \dim \mathcal{N}_1(U).$$

The variety $\mathcal{N}_1(U)$ is irreducible, so $V_{G_1}(Z(w\cdot 0)) = \mathcal{N}_1(U)$.

For any p-regular weight λ, there exist $\nu\in X(T)$ and $w\in W$ such that $\mu = w\cdot 0 + p\nu$ lies in the same alcove as λ. The translation functor T_λ^μ: G_1T-mod $\to$ G_1T-mod carries $\hat{Z}_1(\lambda)$ to $\hat{Z}_1(\mu)$, defining isomorphisms

$$\operatorname{Ext}^\bullet_{G_1T}\left(\hat{Z}_1(\lambda), \bigoplus_{\sigma\in X(T)} \hat{L}_1(\sigma)\right) \cong \operatorname{Ext}^\bullet_{G_1T}\left(\hat{Z}_1(\mu), \bigoplus_{\sigma\in X(T)} \hat{L}_1(\sigma)\right).$$

See [Jan2], II, §9. It follows by [DNP], (7.2) that $c_{G_1T}(\hat{Z}_1(\lambda)) = c_{G_1T}(\hat{Z}_1(\mu))$. Thus, by (2.2.12), $c_{G_1}(Z_1(\lambda)) = c_{G_1}(Z_1(\mu))$, and again $V_{G_1}(Z_1(\lambda)) = \mathcal{N}_1(U)$.

On the other hand, suppose that $V_{G_1}(Z_1(\lambda)) = \mathcal{N}_1(U)$, but that λ is not p-regular. By [Jan1], (4.14)(1), $V_{B_1}(L_1(w\cdot\lambda)) \subsetneqq \mathcal{N}_1(U)$ for all $w\in W$. The module $Z_1(\lambda)$ has a composition series with factors all of the form $L_1(w\cdot\lambda)$ where $w\in W$. Therefore, by (2.2.8),

$$V_{G_1}(Z_1(\lambda)) \subseteqq \bigcup_{w\in W} V_{G_1}(L_1(w\cdot\lambda)) \subsetneqq \mathcal{N}_1(U).$$

This is a contradiction, so λ is p-regular. ☐

For $\lambda\in X(T)$, let $\Gamma_r(\lambda) = W\cdot\lambda + p^m\mathbb{Z}\Phi + p^rX(T)$, where m be the smallest integer such that there exists a root α with $(\lambda+\rho, \alpha^\vee) \notin p^m\mathbb{Z}$. Then $\Gamma_r(\lambda)$ equals the set of weights μ such that $L_r(\mu)$ belongs to the block $\mathcal{B}_r(\lambda)$ in G_r-mod containing $L_r(\lambda)$; see [Jan2], II, (9.19). For $r = 1$, $\Gamma_1(\lambda) = W\cdot\lambda + pX(T)$.

The following result is a variation of the previous theorem.

(4.1.2) Proposition. *Let $p \geqq h$ and $\lambda\in X(T)$.*

(a) *The weight $\lambda\in X(T)_+$ is p-regular if and only if $V_{G_r}(H^0(\lambda)) = V_{G_r}$.*

(b) *If the weight $\lambda\in X(T)$ is not p-regular, then $V_{G_r}(Z_r(\lambda)) \subsetneqq V_{U_r}$.*

Proof. If λ is p-regular, then $p \nmid \dim H^0(\lambda)$. This means that $V_{G_r}(H^0(\lambda)) = V_{G_r}$ [SFB2], (6.1). Conversely, if $V_{G_r}(H^0(\lambda)) = V_{G_r}$, then, by [SFB2], (7.1),

$$V_{G_1}(H^0(\lambda)) = V_{G_1} \cap V_{G_r}(H^0(\lambda)) = V_{G_1}.$$

But, if λ is not p-regular then $V_{G_1}(H^0(\lambda)) \subsetneqq V_{G_1}$ by [Jan1], (4.14)(1). This proves (a).

To see (b), suppose λ is not p-regular, but that $V_{G_r}(Z_r(\lambda)) = V_{U_r}$. By [SFB2], (7.1) again, $V_{G_1}(Z_r(\lambda)) = V_{U_1}$. But $\hat{Z}_r(\lambda)|_{G_1 T}$ is a projective U_1^+-module, so it has a $G_1 T$-module filtration with sections $\hat{Z}_1(\mu)$ for various $\mu \in X(T)$. Since the G_r-compostion factors $L_r(\mu)$ of $Z_r(\lambda)$ satisfy $\mu \in \Gamma_r(\lambda)$ since $L_r(\mu)|_{G_1}$ is a direct sum of copies of $L_1(\mu)$, we see that the G_1-composition factors $L_1(\mu)$ of $Z_r(\lambda)$ all satisfy $\mu \in \Gamma_1(\lambda)$. Thus, if $\hat{Z}_1(\mu)$ appears as a section in the filtration of $\hat{Z}_r(\lambda)|_{G_1 T}$, then $\mu \in \Gamma_1(\lambda)$. Now (4.1.1) implies that $V_{G_1}(Z_r(\lambda)) \subsetneqq V_{U_1}$, a contradiction. □

4.2. Support varieties over Levi subgroups. Let $I \subsetneqq \Pi$ and set $L = L_I$, $P = U_I.L$, and $P^+ = U_I^+.L$.

(4.2.1) Proposition. *If $\lambda \in X(T)$, then $V_{L_r}(Z_r(\lambda)) = V_{L_r}(Z_r^I(\lambda))$.*

Proof. By Mackey theory for $G_r T$ [CPS2], (4.1) and remarks following (1.3.1),

$$\hat{Z}_r(\lambda)|_{P_r^+ T} \cong \mathrm{ind}_{P_r T}^{G_r T}\, \hat{Z}_r^I(\lambda)^e|_{P_r^+ T} \cong \mathrm{ind}_{L_r T}^{P_r^+ T}\, \hat{Z}_r^I(\lambda).$$

Since $\hat{Z}_r^I(\lambda)$ inflates to a $P_r^+ T$-module (which we still denote by $\hat{Z}_r^I(\lambda)$), the tensor identity yields

$$\hat{Z}_r(\lambda)|_{P_r^+ T} \cong \mathrm{ind}_{L_r T}^{P_r^+ T}\, k \otimes \hat{Z}_r^I(\lambda) \cong k[U_r^+] \otimes \hat{Z}_r^I(\lambda).$$

Thus,

$$Z_r(\lambda) \cong k[U_r^+] \otimes Z_r^I(\lambda)$$

as L_r-modules, so $V_{L_r}(Z_r(\lambda)) \subseteq V_{L_r}(Z_r^I(\lambda))$.

To see the reverse inclusion, let $N = k[U_r^+]/k$. The short exact sequence

$$0 \to k \to k[U_r^+] \to N \to 0$$

of $L_r T$-modules splits because the weights of T in N do not lie in $\mathbb{Z}\Phi_I$. Hence, $Z_r^I(\lambda)$ is an L_r-direct summand of $Z_r(\lambda)$, so $V_{L_r}(Z_r^I(\lambda)) \subseteq V_{L_r}(Z_r(\lambda))$. □

4.3. As an application of the above results, we have

(4.3.1) Theorem. *Let $I \subsetneqq \Pi$ and assume that p is good for G.*

(a) $x_I \in V_{(L_I)_1}(Z_1(\lambda))$ if and only if $\Phi_{\lambda,p} \cap \Phi_I = \emptyset$.

(b) $x_I \notin V_{(L_I)_1}\left(\bigoplus_{w \in W} Z_1(w \cdot \lambda)\right)$ if and only if $w(\Phi_{\lambda,p}) \cap \Phi_I \neq \emptyset$ for all $w \in W$.

Proof. We first prove (a). By (4.2.1), we can assume that $L = G$. First, assume that $\Phi_{\lambda,p} = \emptyset$, i.e., λ is p-regular. Then $p \geqq h$ and by (4.1.1), $V_{G_1}(Z_1(\lambda)) = \mathscr{N}_1(U) = \mathscr{N}(G)$, so $x_\Pi \in V_{G_1}(Z_1(\lambda))$. Conversely, suppose that $x_\Pi \in V_{G_1}(Z_1(\lambda)) = V_{B_1}(Z_1(\lambda))$. Since p is good, the B-orbit of x_Π is dense in $\mathscr{N}_1(B)$, so $V_{B_1}(Z_1(\lambda)) = \mathscr{N}_1(U) = \mathscr{N}(U)$. By (4.1.1), $\Phi_{\lambda,p} = \emptyset$ as required. Finally, (b) now follows from (3.4.2). □

4.4. The following theorem provides an upper bound on the support of a module in $\mathscr{B}_r(\lambda)$.

(4.4.1) Theorem. *Let $\lambda \in X(T)$ and let $M \in \mathscr{B}_r(\lambda)$-mod be finite dimensional. Then:*

(a) $V_{B_r}(M) \subseteqq \bigcup_{\mu \in \Gamma_r(\lambda)} V_{G_r}(Z_r(\mu))$.

(b) *If M is a G-module, then* $V_{G_r}(M) \subseteqq \bigcup_{\mu \in \Gamma_r(\lambda)} G \cdot V_{G_r}(Z_r(\mu))$.

Proof. We first prove (a). Let $\Gamma_r'(\lambda) = X_r(T) \cap \Gamma_r(\lambda)$. Given $\mu \in \Gamma_r(\lambda)$, $Z_r(\mu) \cong Z_r(\mu')$ for some $\mu' \in \Gamma_r'(\mu)$. Let $M' = \bigoplus_{\mu \in \Gamma_r'(\lambda)} Z_r(\mu)$ and $M'' = \bigoplus_{\mu \in \Gamma_r'(\lambda)} \mu \in B_r$-mod. Form the commutative diagram

$$\begin{tikzcd}
H(G_r, k) \otimes \operatorname{Ext}^\bullet_{G_r}(M, M') \arrow[r] \arrow[d, "\mathrm{res} \otimes \delta"] & \operatorname{Ext}^\bullet_{G_r}(M, M') \arrow[d, "\sigma"] \\
H(B_r, k) \otimes \operatorname{Ext}^\bullet_{B_r}(M, M'') \arrow[r] & \operatorname{Ext}^\bullet_{B_r}(M, M'')
\end{tikzcd}$$

where δ and σ are isomorphisms implied by the universal mapping property of induction. If $\mathscr{I}$ is the annihilator of $H(G_r, k)$ on $\operatorname{Ext}^\bullet_{G_r}(M, M')$ and $\mathscr{J}$ is the annihilator of $H(B_r, k)$ on $\operatorname{Ext}^\bullet_{B_r}(M, M'')$, then $\mathrm{res}(\mathscr{I}) \subseteqq \mathscr{J}$. Thus, $V_{B_r}(M, M'') = V(\mathscr{J}) \subseteqq V_{G_r}(M, M')$. If $\operatorname{Ext}^\bullet_{B_r}(M, \mu) \neq 0$ for some $\mu \in X(T)$, then $\operatorname{Ext}^\bullet_{G_r}(M, Z_r(\mu)) \neq 0$, so $\mu \in \Gamma_r(\lambda)$. Thus, by (2.2.9), $V(\mathscr{J}) = V_{B_r}(M)$. Hence, (a) follows from (2.2.1) and the containments

$$V_{B_r}(M) = V_{B_r}(M, M'') \subseteqq V_{G_r}(M, M') \subseteqq V_{G_r}(M').$$

Now assume that M is a G-module. Since G acts on $V_{G_r}(M)$ and $G \cdot V_{B_r}(M) = V_{G_r}(M)$ [Bend], (4.5.2)[4], (b) follows also. □

4.5. We will now indicate how Theorem (4.4.1) fits into the context of Jantzen's earlier work. The next corollary is the key proposition ([Jan4], (2.4)) used in his calculation of the supports varieties for $H^0(\lambda)$ in type A.

(4.5.1) Corollary. *Let $r = 1$ and $I \subseteqq \Pi$. If $w(\Phi_{\lambda,p}) \cap \Phi_I \neq \emptyset$ for all $w \in W$, then $x_I \notin V_{G_1}(H^0(\lambda))$.*

Proof. Suppose that $w(\Phi_{\lambda,p}) \cap \Phi_I \neq \emptyset$ for all $w \in W$. Since $\Gamma_1(\lambda) = W \cdot \lambda + pX(T)$, (4.3.1)(a) and (3.4.2) imply that $x_I \notin V_{(L_I)_1}\left(\bigoplus_{\mu \in \Gamma_1(\lambda)} Z_1(\mu)\right)$. Hence, by (4.4.1)(a),

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[4] The reader who does not have access to [Bend] can apply (5.4.1) below with $M = k$ and $t = 0$.

$$x_I \notin V_{B_1}\big(H^0(\lambda)\big) = V_{G_1}\big(H^0(\lambda)\big) \cap V_{B_1},$$

so $x_I \notin V_{G_1}\big(H^0(\lambda)\big)$. □

Let $G = \mathrm{GL}_n(k)$. Any subroot system Φ' of Φ is isomorphic to

$$A_{d_1-1} \times A_{d_2-1} \times \cdots \times A_{d_s-1}$$

where $d_1 \geqq d_2 \geqq \cdots \geqq d_s > 0$ is a partition of n. This defines a bijective correspondence between non-isomorphic subroot systems of Φ and partitions of n. Given $J \subseteqq \Pi$, let $\sigma(J)$ be the partition of n corresponding to the subsystem Φ_J. In addition, given $\lambda \in X(T)$, let $\sigma(\lambda)$ be the partition of n defined by the subroot system $\Phi_{\lambda,p}$ of Φ. For a partition σ of n, let $x_\sigma \in \mathrm{GL}_n(k)$ be the nilpotent matrix in upper triangular Jordan canonical having blocks of the sizes corresponding to the partition σ. Also, let σ^t be the dual partition. From [Kr],

$$\overline{G \cdot x_{\sigma(J)^t}} = G \cdot \mathfrak{u}_J. \tag{4.5.2}$$

We can now recover Jantzen's calculations of the supports of $H^0(\lambda)$ for G_1 where $G = \mathrm{GL}_n$.

(4.5.3) Corollary. *Let $G = \mathrm{GL}_n$. For $\lambda \in X(T)_+$, choose $J \subseteqq \Pi$ such that $\Phi_{\lambda,p} = y(\Phi_J)$ for some $y \in W$. Then $V_{G_1}\big(H^0(\lambda)\big) = G \cdot \mathfrak{u}_J = \overline{G \cdot x_{\sigma(\lambda)^t}}$.*

Proof. The G-stable variety $V_{G_1}\big(H^0(\lambda)\big)$ is a finite union of closures of G-orbits in $\mathcal{N}_1(G)$. Let $x_I \in V_{G_1}\big(H^0(\lambda)\big)$ for some $I \subseteqq \Pi$. By (4.5.1), $w(\Phi_{\lambda,p}) \cap \Phi_I = \emptyset$ for some $w \in W$; equivalently, $w(\Phi_J) \cap \Phi_I = \emptyset$ for some $w \in W$. Consequently, $x_I \in G \cdot \mathfrak{u}_J = \overline{G \cdot x_{\sigma(\lambda)^t}}$ by (1.3.4) and (4.5.2). Hence, $V_{G_1}\big(H^0(\lambda)\big) \subseteqq \overline{G \cdot x_{\sigma(\lambda)^t}}$. By (3.4.3)(b),

$$\dim V_{G_1}\big(H^0(\lambda)\big) \geqq \dim G \cdot \mathfrak{u}_J = |\Phi| - |\Phi_{\lambda,p}|.$$

The variety $\overline{G \cdot x_{\sigma(\lambda)^t}}$ is irreducible, thus $V_{G_1}\big(H^0(\lambda)\big) = \overline{G \cdot x_{\sigma(\lambda)^t}}$. □

4.6. Independence of weights. Fix $r \geqq 1$. For $\lambda \in X(T)_+$, choose $s \geqq r$ so that the composition factors $L(\mu)$ of $H^0(\lambda)$ all satisfy $\mu \in X_s(T)$. Then [Jan2], II, (9.21) implies that each such μ lies in $\Gamma_s(\lambda)$. Since $\Gamma_s(\lambda) \subseteqq \Gamma_r(\lambda)$, we conclude that $H^0(\lambda) \in \mathscr{B}_r(\lambda)$. Thus, (4.4.1)(b) implies that for all $\sigma \in \Gamma_r(\lambda) \cap X(T)_+$, $V_{G_r}\big(H^0(\sigma)\big) \subseteqq G \cdot \left[\bigcup_{\mu \in \Gamma_r(\lambda)} V_{G_r}\big(Z_r(\mu)\big)\right]$.

(4.6.1) Theorem. *Let G be a reductive algebraic group and let $\lambda \in X(T)_+$. The following statements are equivalent:*

(a) $V_{G_r}\big(H^0(\sigma)\big) = G \cdot \left[\bigcup_{\mu \in \Gamma_r(\lambda)} V_{G_r}\big(Z_r(\mu)\big)\right]$ *for all $\sigma \in \Gamma_r(\lambda) \cap X(T)_+$.*

(b) $V_{G_r}\big(H^0(\lambda)\big) = V_{G_r}\big(H^0(\sigma)\big)$ *for all $\sigma \in \Gamma_r(\lambda) \cap X(T)_+$.*

Proof. Clearly (a) ⇒ (b). Conversely, suppose that (b) holds, and let

$$\mathscr{V} = V_{G_r}\big(H^0(\lambda)\big).$$

We claim that $V_{G_r}(L(\mu)) \subseteq \mathscr{V}$ for all $\mu \in \Gamma_r(\lambda) \cap X(T)_+$. If $\mu \in C_{\mathbb{Z}}$, then $L(\mu) = H^0(\mu)$, so the claim holds in this case. Otherwise, form the short exact sequence

$$0 \to L(\mu) \to H^0(\mu) \to N \to 0.$$

If $L(\tau)$ is a composition factor of N, then $\tau < \mu$, so the evident induction and (2.2.8) imply that $V_{G_r}(N) \subseteq \mathscr{V}$. By (2.2.7), $V_{G_r}(L(\mu)) \subseteq \mathscr{V}$. This proves our claim. Given $\mu \in \Gamma_r(\lambda)$, $L_r(\mu) = L(\mu')$ for some $\mu' \in \Gamma_r(\lambda) \cap X(T)_+$. Also, $L(\mu')|_{G_r}$ is a direct sum of copies of $L_r(\mu')$. Thus, by (2.2.7) again, $V_{G_r}(M) \subseteq \mathscr{V}$ for all $M \in \mathscr{B}_r(\lambda)$. In particular, $V_{G_r}\left(\bigoplus_{\mu \in \Gamma_r(\lambda)} Z_r(\mu)\right) \subseteq \mathscr{V}$. Since $\mathscr{V}$ is G-stable, we have $G \cdot \left[\bigcup_{\mu \in \Gamma_r(\lambda)} V_{G_r}(Z_r(\mu))\right] \subseteq V_{G_r}(H^0(\lambda)) = V_{G_r}(H^0(\sigma))$. The reverse inclusion follows by (4.4.1)(b). Thus, (b) ⇒ (a). □

5. Varieties and induction

5.1. Let H be an affine k-group. By an H-algebra, we mean a commutative, finitely generated k-algebra S upon which H acts as rational k-algebra automorphisms. Thus, the structure maps $\nu: S \otimes S \to S$, $a \otimes b \mapsto ab$, and $\eta: k \to S$, $1 \mapsto 1_S$, are morphisms of H-modules. Then a left S-module M is called an $S.H$-module provided that M is a rational H-module and the map $S \otimes M \to M$, $s \otimes m \mapsto sm$, is a morphism of H-modules. Let $S.H$-mod be the category of $S.H$-modules.

(5.1.1) Proposition. *Suppose that M is an $S.H$-module for an affine k-group H and H-algebra S. Suppose H is a closed subgroup of an affine k-group K. Then:*

(a) *The induced module* $\operatorname{ind}_H^K M$ *is an* $(\operatorname{ind}_H^K S).K$*-module.*

(b) *Let R be a K-algebra. For any $R.H$-module M,* $\operatorname{ind}_H^K M$ *has the (unique) structure of an $R.K$-module compatible with the $R.H$-module structure of M. Let I be a K-stable ideal in R which annihilates M. Then I annihilates* $\operatorname{ind}_H^K M$.

(c) *In* (b), *suppose that $R \to S$ is a morphism of H-algebras. Then* ind_H^K *defines a functor* $S.H$-mod $\to$ $R.K$-mod.

Proof. The assertions follow from the universal mapping properties of induction. Let Ev_S: $\operatorname{ind}_H^K S \to S$ be the evaluation map. Then $\nu: S \otimes S \to S$ lifts to a unique map μ of rational K-modules making the following diagram commutative

$$\begin{tikzcd}
\operatorname{ind}_H^K S \otimes \operatorname{ind}_H^K S \arrow[r, "\mu"] \arrow[d, "\mathrm{Ev}_S \otimes \mathrm{Ev}_S"'] & \operatorname{ind}_H^K S \arrow[d, "\mathrm{Ev}_S"] \\
S \otimes S \arrow[r, "\nu"'] & S
\end{tikzcd}$$

and defining a multiplication on $\operatorname{ind}_H^K S$. Similarly, the unit map $\eta: k \to S$ lifts to define a unit map $k \to \operatorname{ind}_H^K S$. It is straightforward to check that the axioms for an (associative) algebra hold for $\mathrm{ind}_H^K S$. In the same way, the module structure map $S \otimes M \to M$ lifts to define the structure of an $\mathrm{ind}_H^K S$-module on $\mathrm{ind}_H^K M$. This proves (a). To see (b), we consider the commutative diagram

$$\begin{tikzcd}
I \otimes \mathrm{ind}_H^K M \arrow[r] \arrow[d] & R \otimes \mathrm{ind}_H^K M \arrow[r] \arrow[d] & \mathrm{ind}_H^K M \arrow[d] \\
I \otimes M \arrow[r] & R \otimes M \arrow[r] & M
\end{tikzcd}$$

obtained by applying ind_H^K to the bottom row. Here we use the fact that, for any K-module V, $\mathrm{ind}_H^K(V \otimes M) \cong V \otimes \mathrm{ind}_H^K M$. The right-hand square in the above diagram proves that $\mathrm{ind}_H^K M$ has the unique structure of an $R.K$-module which is compatible with the $R.H$-module structure of M. Since I annihilates M, the universal mapping property of induction implies that I annihilates $\mathrm{ind}_H^K M$. Finally, (c) is clear. □

5.2. Let H be a connected algebraic group over k, and let S be an H-algebra. It will be convenient to recast the category $S.H$-mod in terms of the module category of a certain algebra of "differential operators". Thus, let $\mathrm{Dist}(H)$ be the distribution Hopf algebra of H [DG], II; §4, (6.1), [T], Ch. 3. Let $\mathfrak{f}$: H-mod $\to$ $\mathrm{Dist}(H)$-mod be the natural functor which assigns to any rational B-module its associated $\mathrm{Dist}(B)$-module structure. Then $\mathfrak{f}$ is a full embedding, and so defines an equivalence of categories from H-mod to its strict image $\mathrm{Dist}(H)$-mod$'$ in $\mathrm{Dist}(H)$-mod.

Since $\eta\colon k \to S$ and $\nu\colon S \otimes S \to S$ are H-module morphisms, they induce morphisms of $\mathrm{Dist}(H)$-modules. Equivalently, S is a left "module algebra" for $\mathrm{Dist}(H)$, in the sense that, for $h \in \mathrm{Dist}(B)$, $a, b \in S$, we have

$$\begin{cases} h(ab) = \sum (h_{(1)}a)(h_{(2)}b), \\ h1_S = \varepsilon(h)1_S \end{cases}$$

if $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ is the image of h under the comultiplication

$$\Delta\colon \mathrm{Dist}(H) \to \mathrm{Dist}(H) \otimes \mathrm{Dist}(H)$$

and $\varepsilon\colon \mathrm{Dist}(H) \to k$ is the counit on $\mathrm{Dist}(H)$.

Let $D = S \# \mathrm{Dist}(H)$ be the smash product of S and $\mathrm{Dist}(H)$ [M], (4.1.1). As a vector space, $D = S \otimes \mathrm{Dist}(H)$, with multiplication given by

$$(a \otimes h)(a' \otimes h') = \sum a(h_{(1)}a') \otimes h_{(2)}h'.$$

A D-module $\mathscr{M}$ is merely a module simultaneously for both S and $\mathrm{Dist}(H)$ such that

$$h(am) = \sum (h_{(1)}a)(h_{(2)}m)$$

for all $h \in \mathrm{Dist}(H)$, $a \in S$, and $m \in \mathscr{M}$.

Consider the functor

$$\mathfrak{F}\colon \mathrm{Dist}(H)\text{-mod} \to D\text{-mod}$$

defined by $\mathfrak{F}(-) = D \otimes_{\mathrm{Dist}(H)} -$. Alternatively, for $M \in \mathrm{Dist}(H)$-mod, $\mathfrak{F}(M) = S \otimes M$ as vector spaces with action as follows: for $a, b \in S$, $h \in \mathrm{Dist}(H)$, and $m \in M$, we have

$$a(b \otimes m) = ab \otimes m; \quad h(b \otimes m) = \sum h_{(1)}b \otimes h_{(2)}m.$$

The functor $\mathfrak{F}$ is exact and provides a left adjoint to the natural restriction functor

$$\mathfrak{G}\colon D\text{-mod} \to \mathrm{Dist}(H)\text{-mod}$$

defined by restricting any D-module to the subalgebra $\mathrm{Dist}(H)$ of D. Thus, $\mathfrak{G}$ takes injective D-modules to injective $\mathrm{Dist}(H)$-modules. Given $M \in D$-mod, we often denote $\mathfrak{G}(M)$ simply by just M again.

If D-mod$'$ is the full subcategory of D-mod with objects $\mathscr{V}$ satisfying $\mathfrak{G}(\mathscr{V}) \in \mathrm{Dist}(H)$-mod$'$, then D-mod$'$ is equivalent to the category $S.H$-mod. Since S is a rational H-module, the functor $\mathfrak{F}$ carries $\mathrm{Dist}(H)$-mod$'$ to D-mod$'$ and it provides a left adjoint to $\mathfrak{G}$ (restricted to D-mod$'$). The definitions imply that if $\psi\colon \mathscr{V} \to \mathscr{W}$ is a morphism in D-mod with $\mathscr{V} \in D$-mod$'$, then the image $\psi(\mathscr{V})$ is a submodule of $\mathscr{W}$ lying in D-mod$'$.

The category D-mod$'$ contains enough injectives. In fact, given $\mathscr{V} \in D$-mod$'$, let $\tilde{\mathscr{I}}$ be a D-injective module containing $\mathscr{V}$ (which exists because D-mod is the module category for a ring). Let $\mathscr{I}$ be the sum (or union) of all submodules of $\tilde{\mathscr{I}}$ which lie in D-mod$'$. Then $\mathscr{V} \subseteq \mathscr{I}$. From remarks at the end of the previous paragraph, it follows directly that $\mathscr{I}$ is an injective in the category D-mod$'$. Also, $\mathfrak{G}\colon D\text{-mod}' \to \mathrm{Dist}(H)\text{-mod}'$ carries injectives to injectives.

Now let $H = B$ be the fixed Borel subgroup in the reductive group G.[5] Let R be a G-algebra and let $R \to S$ be a B-algebra morphism. For $\mathscr{M} \in D$-mod$'$, let $0 \to \mathscr{M} \to \mathscr{I}^\bullet$ be an injective resolution in D-mod$'$ (and hence an injective resolution in B-mod). By (5.1.1)(c), $0 \to \mathrm{ind}_B^G \mathscr{M} \to \mathrm{ind}_B^G \mathscr{I}^\bullet$ is a complex of $R.G$-modules so that, for $n \geqq 0$, $R^n \mathrm{ind}_B^G \mathscr{M}$ inherits an $R.G$-module structure which is functorial in $\mathscr{M}$.

The next result relates the annihilators of the higher right derived functors $R^n \mathrm{ind}_B^G \mathscr{M}$ with the annihilator on $\mathscr{M}$. Recall the notation of (1.3.2): if $\mathscr{I}$ is an ideal in a G-algebra R, we let $\ell_G(\mathscr{I})$ be the largest G-stable ideal of R contained in $\mathscr{I}$.

(5.2.1) Proposition. *Let R be a G-algebra and let S be a B-algebra. Let $\pi^*\colon R \to S$ be a B-algebra morphism. For $\mathscr{M} \in S \# \mathrm{Dist}(B)$-mod$' \cong S.B$-mod and $n \geqq 0$, we have*

$$\ell_G(\pi^{*-1}(\mathrm{Ann}_S \mathscr{M})) \subseteq \mathrm{Ann}_R R^n \mathrm{ind}_B^G \mathscr{M}.$$

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[5] In this case, the category $\mathrm{Dist}(B)$-mod$'$ can be explicitly described. Locally finite modules for the distribution algebra $\mathrm{Dist}(T)$ of the torus T are direct sums of one-dimensional submodules defined by elements in the character group $X(\mathrm{Dist}(T))$. Now $X(T) \subseteq X(\mathrm{Dist}(T))$, and $\mathrm{Dist}(B)$-mod$'$ is the full subcategory of $\mathrm{Dist}(B)$-mod with objects consisting of those $\mathscr{V}$ which are $\mathrm{Dist}(B)$-locally finite and which have $\mathrm{Dist}(T)$-weights lying in $X(T)$. See [CPS1], §9.4 for details.

Proof. By (5.1.1)(b), $\ell_G(\pi^{*-1}(\mathrm{Ann}_S \mathscr{M}))$ annihilates $\mathrm{ind}_B^G \mathscr{M}$. Now $\mathscr{M} \in D$-mod$'$, where $D = S/\mathrm{Ann}_S \mathscr{M} \# \mathrm{Dist}(B)$. Consider a short exact sequence

$$(5.2.2) \qquad 0 \to \mathscr{M} \to \mathscr{I} \to \mathscr{L} \to 0,$$

in D-mod$'$ in which $\mathscr{I}$ is injective. By construction, $\mathrm{Ann}_S \mathscr{M} \subseteqq \mathrm{Ann}_S \mathscr{T}$ for $\mathscr{T} = \mathscr{M}, \mathscr{I}, \mathscr{L}$. Because $\mathscr{M}$, $\mathscr{I}$, and $\mathscr{L}$ are rational B-modules, (5.2.2) determines an exact sequence of $R.G$-modules

$$(5.2.3) \qquad 0 \to \mathrm{ind}_B^G \mathscr{M} \to \mathrm{ind}_B^G \mathscr{I} \to \mathrm{ind}_B^G \mathscr{L} \to R^1 \mathrm{ind}_B^G \mathscr{M} \to 0$$

as well as $R.G$-module isomorphisms

$$(5.2.4) \qquad R^n \mathrm{ind}_B^G \mathscr{L} \cong R^{n+1} \mathrm{ind}_B^G \mathscr{M}, \quad \forall n \geqq 1.$$

By (5.2.3) and (5.1.1)(b), $\ell_G(\pi^{*-1}(\mathrm{Ann}_S \mathscr{M})) \subseteqq \mathrm{Ann}_R R^1 \mathrm{ind}_B^G \mathscr{M}$, while (5.2.4) and induction imply that

$$\begin{aligned} \ell_G(\pi^{*-1}(\mathrm{Ann}_S \mathscr{M})) &\subseteqq \ell_G(\pi^{*-1}(\mathrm{Ann}_S \mathscr{L})) \\ &\subseteqq \mathrm{Ann}_R R^n \mathrm{ind}_B^G \mathscr{L} \\ &= \mathrm{Ann}_R R^{n+1} \mathrm{ind}_B^G \mathscr{M} \end{aligned}$$

for all $n \geqq 1$. □

5.3. Now fix an affine k-group G. In the next section, we will work with a pull-back diagram of closed subgroups of G:

(5.3.1)

$$\begin{tikzcd} K \arrow[r, "\lhd"] & G \\ H \cap K \arrow[u] \arrow[r, "\lhd"'] & H \arrow[u] \end{tikzcd}$$

in which we have written $H \cap K$ for $H \times_G K$. Here K (resp. $H \cap K$) is a normal subgroup of G (resp. H). Now let M be a rational G-module, and consider the commutative diagram

(5.3.2)

$$\begin{tikzcd} G\text{-mod} \arrow[r, "\mathrm{Hom}_K(M,-)"] & G/K\text{-mod} \\ H\text{-mod} \arrow[u, "\mathrm{ind}_H^G"] \arrow[r, "\mathrm{Hom}_{H \cap K}(M,-)"'] & H/H \cap K\text{-mod} \arrow[u, "\mathrm{ind}_{H/H \cap K}^{G/K}"'] \end{tikzcd}$$

of functors. Write

$$\mathscr{F}(M,-) = \mathrm{Hom}_K(M, \mathrm{ind}_H^G -) = \mathrm{ind}_{H/H \cap K}^{G/K} \mathrm{Hom}_{H \cap K}(M,-)$$

for the composite functors. For $N \in H$-mod N, this gives two spectral sequences

$$(5.3.3) \qquad \hat{E}_2^{m,n}(M,N) = \mathrm{Ext}_K^m(M, R^n \operatorname{ind}_H^G N) \Rightarrow (R^{m+n}\mathcal{F})(M,N),$$

$$(5.3.4) \qquad E_2^{m,n}(M,N) = R^m \operatorname{ind}_{H/H\cap K}^{G/K} \mathrm{Ext}^n_{H\cap K}(M,N) \Rightarrow (R^{m+n}\mathcal{F})(M,N)$$

both converging to $R^\bullet\mathcal{F}(M,N)$; see [Jan2], I, (6.12). Also, given M, N above, for any $r \geqq 2$, $E_r^{\bullet,\bullet}(M,N)$ is a differential (graded) module for $E_r^{\bullet,\bullet}(k,k)$.

5.4. Fix $r \geqq 1$ and set $R = H(G_r, k)$ and $S = H(B_r, k)$ for some fixed integer $r \geqq 1$. Let $\pi^*\colon R \to S$ be the restriction map on cohomology. The morphism $\pi\colon V_{B_r} \to V_{G_r}$ induced by π^* maps V_{B_r} homeomorphically onto its image. As discussed in (2.2.10), we will always identify V_{B_r} with its image in V_{G_r}.

(5.4.1) Theorem. *Let M be a rational B-module such that $R^m \operatorname{ind}_B^G M = 0$ for $m \neq t$, where t is some fixed integer. Then*

$$V_{G_r}(R^t \operatorname{ind}_B^G M) = G \cdot V_{B_r}(R^t \operatorname{ind}_B^G M, M).$$

Proof. By (2.2.1), $V_{B_r}(R^t \operatorname{ind}_B^G M, M) \subseteqq V_{B_r}(R^t \operatorname{ind}_B^G M) \subseteqq V_{G_r}(R^t \operatorname{ind}_B^G M)$, so that

$$G \cdot V_{B_r}(R^t \operatorname{ind}_B^G M, M) \subseteqq V_{G_r}(R^t \operatorname{ind}_B^G M).$$

We consider the spectral sequences (5.3.3) and (5.3.4) for $(H,K) = (B, G_r)$ and $(M,N) = (R^t \operatorname{ind}_B^G M, M)$. Since $R^n \operatorname{ind}_B^G M = 0$ for all $n \neq t$, the spectral sequence (5.3.3) collapses, so we finally obtain a spectral sequence:

$$E_2^{m,n} = R^m \operatorname{ind}_{B/B_r}^{G/G_r} \mathrm{Ext}^n_{B_r}(R^t \operatorname{ind}_B^G M, M) \Rightarrow \mathrm{Ext}_{G_r}^{m+n-t}(R^t \operatorname{ind}_B^G M, R^t \operatorname{ind}_B^G M).$$

Let $\mathcal{M} = \mathrm{Ext}^\bullet_{B_r}(R^t \operatorname{ind}_B^G M, M)$. By (5.2.1), the ideal $\mathcal{I} = \ell_G(\pi^{*-1}(\mathrm{Ann}_S \mathcal{M}))$ of $R = H(G_r, k)$, which defines

$$G \cdot V(\mathrm{Ann}_S \mathcal{M}) = G \cdot V_{B_r}(R^t \operatorname{ind}_B^G M, M),$$

annihilates $E_2^{m,n} = R^m \operatorname{ind}_{B/B_r}^{G/G_r} \mathcal{M}$ for all non-negative integers m. Thus, $\mathcal{I} \subseteqq \mathrm{Ann}_R E_\infty^{\bullet,\bullet}$. Since $R^m \operatorname{ind}_{B/B_r}^{G/G_r} = 0$ for $m > N = \dim G/B$, $\mathrm{Ext}^\bullet_{G_r}(R^t \operatorname{ind}_B^G M, R^t \operatorname{ind}_B^G M)$ has a finite increasing R-stable filtration $0 = F^0 \subseteqq F^1 \subseteqq \cdots \subseteqq F^N = \mathrm{Ext}^\bullet_{G_r}(R^t \operatorname{ind}_B^G M, R^t \operatorname{ind}_B^G M)$ in which $F^i/F^{i-1} = \bigoplus_j E_\infty^{N-i+1,j}$. Since $\mathcal{I} F^i \subseteqq F^{i-1}$, $\mathcal{I}^N$ annihilates

$$\mathrm{Ext}^\bullet_{G_r}(R^t \operatorname{ind}_B^G M, R^t \operatorname{ind}_B^G M).$$

Thus, $V_{G_r}(R^t \operatorname{ind}_B^G M) \subseteqq G \cdot V_{B_r}(R^t \operatorname{ind}_B^G M, M)$, as required. □

5.5. There are several applications of Theorem (5.4.1) worth mentioning. The first gives a characterization of $V_{G_r}(H^0(\lambda))$ when $\lambda \in X_+(T)$ in terms of relative support varieties.

(5.5.1) Corollary. *Let G a reductive algebraic group. For $\lambda \in X(T)_+$,*

$$V_{G_r}(H^0(\lambda)) = G \cdot V_{B_r}(H^0(\lambda), \lambda).$$

Now let $\bar{C}_{\mathbb{Z}} = \bar{C} \cap X(T)$ be the set of integral weights in the closure of the bottom p-alcove. We can prove an inclusion of supports for $H^0(\lambda)$ for $\lambda \in \bar{C}_{\mathbb{Z}}$.

(5.5.2) Corollary. *Let $\lambda \in \bar{C}_{\mathbb{Z}}$ and $w \in W$. Then*

$$V_{G_r}(L(\lambda)) = V_{G_r}(H^0(\lambda)) \subseteqq G \cdot V_{G_r}(Z_r(w \cdot \lambda)).$$

Proof. For $w \in W$, $L(\lambda) = H^0(\lambda) = H^{l(w)}(w \cdot \lambda)$. Furthermore, $H^i(w \cdot \lambda) = 0$ for $i \neq l(w)$. Therefore, by (5.4.1), (2.3.1)(c), and finally (2.2.1), we obtain

$$\begin{aligned} V_{G_r}(H^{l(w)}(w \cdot \lambda)) &= G \cdot V_{B_r}(H^{l(w)}(w \cdot \lambda), w \cdot \lambda) \\ &= G \cdot V_{G_r}(H^{l(w)}(w \cdot \lambda), Z_r(w \cdot \lambda)) \\ &\subseteqq G \cdot V_{G_r}(Z_r(w \cdot \lambda)), \end{aligned}$$

as required. □

5.6. The following key result generalizes Corollary (5.5.2).

(5.6.1) Theorem. *Let $\lambda \in X(T)_+$. Then for all $w \in W$,*

(a) $V_{G_1}(L(\lambda)) \subseteqq G \cdot V_{G_1}(Z_1(w \cdot \lambda))$;

(b) $V_{G_1}(H^0(\lambda)) \subseteqq G \cdot V_{G_1}(Z_1(w \cdot \lambda))$.

Proof. If (a) holds, then (b) follows by the linkage principle, together with (2.2.7), so we prove (a). We will use induction on the ordering of dominant weights to prove that

$$V_{G_1}(L(\lambda)) \subseteqq G \cdot V_{G_1}(Z_1(w \cdot \lambda)) \tag{5.6.2}$$

for all $w \in W$. For $\lambda \in \bar{C}_{\mathbb{Z}}$, the inclusion (5.6.2) follows by (5.5.2). Now let $\lambda \in X(T)_+$ be fixed and assume that (5.6.2) holds whenever λ is replaced by any dominant weight μ satisfying $\mu < \lambda$. By using (2.2.7), it suffices to prove that

$$V_{G_1}(H^{l(w)}(w \cdot \lambda)) \subseteqq G \cdot V_{G_1}(Z_1(w \cdot \lambda)) \tag{5.6.3}$$

because $L(\lambda)$ is a composition factor of $H^{l(w)}(w \cdot \lambda)$ of multiplicity one and all other composition factors $L(\mu)$ have the property that $\mu < \lambda$ and μ is linked to λ [Jan2], II, (6.16). Here we are using the fact that since λ and μ are linked (i.e., W_p-conjugate), there exists $x \in W$ such that $Z_1(x \cdot \mu) \cong Z_1(w \cdot \lambda)$.

In order to prove (5.6.3), it suffices to show that if $\mathfrak{m}$ is any maximal ideal of $R = H(G_1, k)$ not contained in the support of $G \cdot V_{G_1}\big(Z_1(w \cdot \lambda)\big)$, then $\mathfrak{m}$ is not in the support of the R-module $\operatorname{Ext}^{\bullet}_{G_1}\big(H^{l(w)}(w \cdot \lambda), H^{l(w)}(w \cdot \lambda)\big)$, i.e.,

$$(5.6.4) \qquad \operatorname{Ext}^{\bullet}_{G_1}\big(H^{l(w)}(w \cdot \lambda), H^{l(w)}(w \cdot \lambda)\big)_{\mathfrak{m}} = 0$$

(see [Jac], Prop. 7.5). In the following, fix this maximal ideal $\mathfrak{m}$.

Applying (5.3.3) and (5.3.4) for $(H, K) = (B, G_1)$ and $(M, N) = \big(H^{l(w)}(w \cdot \lambda), w \cdot \lambda\big)$ yields spectral sequences

$$(5.6.5) \quad \hat{E}_2^{m,n} = \operatorname{Ext}^m_{G_1}\big(H^{l(w)}(w \cdot \lambda), H^n(w \cdot \lambda)\big) \Rightarrow (R^{m+n}\mathscr{F})(M, N),$$

$$(5.6.6) \quad E_2^{m,n} = R^m \operatorname{ind}_{B/B_1}^{G/G_1} \operatorname{Ext}^n_{B_1}\big(H^{l(w)}(w \cdot \lambda), w \cdot \lambda\big) \Rightarrow (R^{m+n}\mathscr{F})(M, N).$$

For any given $r \geqq 2$, $\hat{E}_r^{\bullet,\bullet}$ is a differential module for the differential algebra $\hat{E}_r^{\bullet,\bullet}$ obtained from the spectral sequence (5.3.3) obtained by taking $M = N = k$. But

$$\hat{E}_2^{\bullet,\bullet}(k, k) = \hat{E}_2^{\bullet,0}(k, k) = \hat{E}_\infty^{\bullet,\bullet}(k, k) = \hat{E}_\infty^{\bullet,0}(k, k) \cong H^{\bullet}(G_1, k),$$

so (5.6.5) is a spectral sequence of R-modules. Similarly, from remarks after (5.3.4), it follows that the edge map $R \to E_r^{\bullet,\bullet}$, $r \geqq 2$, gives the spectral sequence (5.6.6) the structure of an R-module.

By (5.2.1) and (2.3.1)(c)[6],

$$V(\operatorname{Ann}_R E_2) \subseteqq G \cdot V_{B_1}\big(H^{l(w)}(w \cdot \lambda), w \cdot \lambda\big) \subseteqq G \cdot V_{G_1}\big(Z_1(w \cdot \lambda)\big).$$

Hence, the localization $(E_2)_{\mathfrak{m}}$ vanishes identically. Because localization at $\mathfrak{m}$ is an exact functor, it now follows that $(E_\infty)_{\mathfrak{m}} = 0$, too. Thus, $\big((R^{\bullet}\mathscr{F})(M, N)\big)_{\mathfrak{m}} = 0$ and, finally,

$$(5.6.7) \qquad (\hat{E}_\infty)_{\mathfrak{m}} = 0.$$

We consider the spectral sequence $\{\tilde{E}_r, \tilde{d}_r\} = \{(\hat{E}_r)_{\mathfrak{m}}, \tilde{d}_r\}$ obtained by localizing (5.6.5) at the ideal $\mathfrak{m}$. For any integer r, $\tilde{E}_r$ is a graded R-module with

$$(5.6.8) \qquad \tilde{E}_r^s = \Big(\bigoplus_t \hat{E}_r^{t,s}\Big)_{\mathfrak{m}},$$

while the differential $\tilde{d}_r$ has degree $-r + 1$. For $n \neq l(w)$, the composition factors of $H^n(w \cdot \lambda)$ have high weights which are both linked to and strictly less than λ [Jan2], II, (6.15), so by our induction hypothesis $V_{G_1}\big(H^n(w \cdot \lambda)\big) \subseteqq G \cdot V_{G_1}\big(Z_1(w \cdot \lambda)\big)$. We conclude from (2.2.1) and (5.6.8) that $\tilde{E}_2^n = 0$ for all $n \neq l(w)$. Thus, $\tilde{E}_2$ is concentrated in degree $l(w)$, so we must have $\tilde{E}_2 = 0$, since, by (5.6.7), $\tilde{E}_\infty = (\hat{E}_\infty)_{\mathfrak{m}} = 0$. In particular, (5.6.4) holds. □

(5.6.9) Remark. When $r > 1$, the inductive step laid out in the first paragraph of the proof fails, since, even if μ is linked to λ there, there may not exist $x \in W$ such that

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[6] The remaining part of the argument would appear to be close to that intended in [Ost], (5.1).

$Z_r(x \cdot \mu) \cong Z_r(w \cdot \lambda)$, or even such that $V_{G_r}(Z_r(w \cdot \mu)) = V_{G_r}(Z_r(w \cdot \lambda))$. However, the proof can be easily modified to give the following equality

$$V_{G_r}(H^0(\lambda)) = G \cdot \left[\bigcup_{\mu \uparrow \lambda, \mu \in X(T)_+} V_{G_r}(Z_r(w \cdot \mu)) \right] \tag{5.6.10}$$

for any $w \in W$. Here $\uparrow$ is as in [Jan2], II, §6.

6. Applications

Throughout this section, G will be a fixed reductive group over k. We will identify V_{G_1} with its image in $\mathcal{N}_1(G)$; see the discussion around (2.2.10).

6.1. The following result provides a condition which guarantees that $V_{G_r}(Z_r(\lambda))$ lies in the nilpotent radical $\mathfrak{u}_I$ of a parabolic subalgebra.

(6.1.1) Proposition. *Let L_I be a Levi subgroup of G corresponding to $I \subseteqq \Pi$. Let $r \geqq 1$ and $\lambda \in X(T)$ be given such that $Z_r^I(\lambda)$ is a projective L_r-module (i.e., $(\lambda + \rho, \alpha^\vee) \in p^r\mathbb{Z}$ for all $\alpha \in I$). Then*

$$V_{G_r}(Z_r(\lambda)) \subseteqq V_{(U_I)_r}.$$

When $r = 1$, $V_{(U_I)_1} \subseteqq \mathfrak{u}_I$.

Proof. In this proof, we write L_r for $(L_I)_r$, P_r for $(P_I)_r$, and U_r for $(U_I)_r$. If $Z_r^I(\lambda)^e$ denotes the inflation of $Z_r^I(\lambda)$ from L_r to P_r, then $Z_r(\lambda) = \operatorname{ind}_{P_r}^{G_r} Z_r^I(\lambda)^e$, as noted in (1.3). By (2.3.1)(a), $V_{G_r}(Z_r(\lambda)) \subseteqq V_{P_r}(Z_r^I(\lambda)^e)$. For $M = Z_r^I(\lambda)^e$, consider the commutative diagram

$$\begin{tikzcd} H(P_r,k) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & H(U_r,k) \arrow[d, "\delta"] \\ \operatorname{Ext}^\bullet_{P_r}(M,M) \arrow[r, "\beta"'] & \operatorname{Ext}^\bullet_{U_r}(M,M) \end{tikzcd} \tag{6.1.2}$$

in which α, β are the natural restriction maps on cohomology, while $\gamma = - \otimes M$ and $\delta = - \otimes M$. Because the action of U_r on $M = Z_r^I(\lambda)^e$ is trivial,

$$\operatorname{Ext}^\bullet_{U_r}(M,M) \cong \operatorname{Ext}^\bullet_{U_r}(k,k) \otimes M^* \otimes M$$

and δ is an injection. On the other hand, because M is projective for L_r, the Hochschild-Serre spectral sequence for $\operatorname{Ext}^\bullet_{P_r}$, using the normal subgroup U_r of P_r and quotient group $L_r = P_r/U_r$, shows that β maps $\operatorname{Ext}^\bullet_{P_r}(M,M)$ isomorphically into its image $\operatorname{Ext}^\bullet_{U_r}(M,M)^{L_r}$. Therefore, $\ker(\alpha) = \ker(\gamma)$. Since $V_{P_r}(Z_r^I(\lambda)^e) = V(\ker(\gamma))$ and $\operatorname{Im}(V_{U_r} \to V_{P_r}) = V(\ker(\alpha))$, we conclude that $V_{G_r}(Z_r(\lambda)) \subseteqq V_{P_r}(Z_r^I(\lambda)^e) = V_{U_r}$, identifying V_{U_r} with its image in V_{G_r}. $\square$

We have the following consequence when $r = 1$.

(6.1.3) Corollary. *For $\lambda \in X(T)_+$ and p good, choose $I \subseteq \Pi$ so that $w(\Phi_{\lambda,p}) = \Phi_I$ for some $w \in W$. Then*

$$V_{G_1}\bigl(H^0(\lambda)\bigr) \subseteqq G \cdot V_{(U_I)_1}.$$

Proof. By (5.6.1), $V_{G_1}\bigl(H^0(\lambda)\bigr) \subseteqq G \cdot V_{G_1}\bigl(Z_1(w \cdot \lambda)\bigr)$. By (3.4.2), $\Phi_{w \cdot \lambda, p} = \Phi_I$, so by (6.1.1), $V_{G_1}\bigl(Z_1(w \cdot \lambda)\bigr) \subseteqq V_{(U_I)_1}$. □

6.2. Jantzen conjecture. As another application of our results, we answer affirmatively and completely the question raised by Jantzen [Jan4], (2.7)(1). First, recall the definition of $\Phi_{\lambda,p}$ for $r = 1$ given in (3.4.2). For any $\alpha \in \Phi$, suppose $\alpha = \sum_{i=1}^{l} n_i \alpha_i$, where $n_i \in \mathbb{Q}$. Then $d_\alpha \alpha^\vee = \sum n_i d_i \alpha_i^\vee$. In particular, this implies that if $\alpha, \beta \in \Phi$ satisfy $\alpha + \beta \in \Phi$, then $d_{\alpha+\beta}(\alpha + \beta)^\vee = d_\alpha \alpha^\vee + d_\beta \beta^\vee$. It follows that $\Phi_{\lambda,p}$ is a (integrally) closed subroot system of Φ. Moreover, $\Phi_{\lambda,p}^\vee$ is also a closed subroot system of the dual root system $\Phi^\vee$.

In addition, if p is good, then $\Phi_{\lambda,p} = \{\alpha \in \Phi \mid (\lambda + \rho, \alpha^\vee) \in p\mathbb{Z}\}$. The definition of "good prime" now implies that $\Phi_{\lambda,p} = \mathbb{Q}\Phi_{\lambda,p} \cap \Phi$, so, by [B], Prop. 24, p. 165, there exists a subset $I \subseteqq \Phi$ and an element $w \in W$ such that $\Phi_I = w(\Phi_{\lambda,p})$. (See also [Jan4], (2.7).)

(6.2.1) Theorem. *Let G be a reductive algebraic group and assume that p is good. Let $\lambda \in X(T)_+$. Choose $w \in W$ such that $w(\Phi_{\lambda,p}) = \Phi_I$ for some $I \subseteqq \Pi$. Then $V_{G_1}\bigl(H^0(\lambda)\bigr) = G \cdot \mathfrak{u}_I$.*

Proof. By (6.1.3), $V_{G_1}\bigl(H^0(\lambda)\bigr) \subseteqq G \cdot V_{(U_I)_1}$. Under our identifications, $V_{(U_I)_1}$ is identified with the subvariety $\mathcal{N}_1(U_I)$ of $\mathcal{N}(U_I) = \mathfrak{u}_I$. By (3.4.3)(b) and (1.3.3),

$$c_{G_1}\bigl(H^0(\lambda)\bigr) \geqq |\Phi| - |\Phi_I| = \dim G \cdot \mathfrak{u}_I.$$

The result follows by the irreducibility of $G \cdot \mathfrak{u}_I$. □

As an immediate consequence, we have the following result. Compare (3.4.3)(b).

(6.2.2) Corollary. *Let G be a reductive algebraic group and assume that p is good. For $\lambda \in X(T)_+$, we have*

$$c_{G_1}\bigl(H^0(\lambda)\bigr) = |\Phi| - |\Phi_{\lambda,p}|.$$

6.3. Taking $\lambda = 0$ in the above theorem gives the following result, since V_{G_1} is homeomorphic to $\mathcal{N}_1(G)$. This partially answers a question raised in [FP1], (3.4). Recall the notation $d(\Phi, p) = |\Phi_{0,p}|$.

(6.3.1) Corollary. *Assume that p is a good prime for the reductive group G. The variety $\mathcal{N}_1(G)$ is irreducible of dimension $|\Phi| - d(\Phi, p)$.*

It is unknown (to the authors at least) whether $\mathcal{N}_1(G)$ is irreducible in general. However, even when p is a bad prime, the methods of this paper do give information on $\dim \mathcal{N}_1(G)$. For example, suppose that G has type G_2. In case $p = 3$, $\Phi_{0,3}$ consists of all long roots in Φ. In this case, we can modify the argument in Section 3, replacing the cyclotomic polynomial $\Psi_3(t)$ by $\Psi_9(t)$, to conclude that

$$\dim \mathcal{N}_1(G) \geqq |\Phi| - |\{\alpha \in \Phi \mid (\rho, \alpha^\vee) \in 3\mathbb{Z}\}| = 10.$$

For $x, y \in \mathfrak{g}$,

$$(x+y)^{[3]} = x^{[3]} + [[y, x], x] + [[x, y], y] + y^{[3]}.$$

For any root α, $\mathfrak{g}_\alpha^{[3]} = 0$. Thus, using [St], Satz 3′, it follows that $\mathcal{N}_1(G)$ is the union of five G-conjugacy classes defined by the elements $u \in \{x_\beta + x_{2\alpha+\beta}, x_\beta + x_{\alpha+\beta}, x_\alpha, x_\beta, 0\}$, where α (resp., β) is the simple short (resp., long) root. For such u, it is easy to determine that $\dim Z_G(u) \geqq 4$, so we conclude that $\dim \mathcal{N}_1(G) = 10$; in fact, it must be the closure of the G-orbit of $x_\beta + x_{2\alpha+\beta}$.[7] In case $p = 2$, we have

$$(x+y)^{[2]} = x^{[2]} + [x, y] + y^{[2]}.$$

Using [St], Satz 2′, $\mathcal{N}_1(G) = \overline{G \cdot x_\beta}$, where β is a short root. Now

$$\dim \mathcal{N}_1(G) = 8 = |\Phi| - d(\Phi, 2),$$

and the estimate is exact.

6.4. Now assume that p is good. Since $\mathcal{N}_1(G)$ is an irreducible variety and since G has only finitely many orbits in $\mathcal{N}(G)$, $\mathcal{N}_1(G)$ contains a dense, open orbit $\mathcal{O}_{[p]\text{-reg}}$. Any element $x \in \mathcal{O}_{[p]\text{-reg}}$ is called a $[p]$-regular element.

(6.4.1) Corollary. *Let G be a reductive group and assume p is good. Let $\lambda \in X(T)_+$ be such that $|\Phi_{\lambda,p}| > d(\Phi, p)$. Then the G-modules $H^0(\lambda)$ and $L(\lambda)$ are x-projective modules for any $x \in \mathcal{O}_{[p]\text{-reg}}$.*

Proof. Suppose that $w(\Phi_{\lambda,p}) = \Phi_I$. Then

$$\dim G \cdot \mathfrak{u}_I = |\Phi| - |\Phi_{\lambda,p}| < \dim \mathcal{N}_1(G).$$

Thus, if $x \in \mathcal{O}_{[p]\text{-reg}}$, then x does not belong to $V_{G_1}(H^0(\lambda))$. It follows that $H^0(\lambda)$ is x-projective. If $\mu \in X(T)_+$ satisfies $\mu = w \cdot \lambda$ for some $w \in W_p$, then $|\Phi_{\mu,p}| = |\Phi_{\lambda,p}| > d(\Phi, p)$. Since $H^0(\lambda) = L(\lambda)$ if $\lambda \in \bar{C}_{\mathbb{Z}}$, the evident induction on λ proves that $L(\lambda)$ is x-projective also. □

When $p \geqq h$, the condition that $|\Phi_{\lambda,p}| > d(\Phi, p)$ amounts to saying that λ is not p-regular. Thus, the result above extends [Jan1], (4.14)(1).

6.5. It is interesting to observe that the results above can be used to give "homological proofs" of several well known results on nilpotent (and unipotent) classes which are contained in the restricted nullcone $\mathcal{N}_1(G)$. In particular, if $p \geqq h$, then these proofs apply to $\mathcal{N}(G)$. We give two examples, assuming for simplicity that $p \geqq h$.

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[7] Although [St], Satz 3′ gives no indication of the orbit closures, they are easy to work out. First, conjugating $x_\beta + x_{2\alpha+\beta}$ by root group elements $u_{-\alpha}(t)$ and then by a torus element yields, in characteristic 3, that $x_\beta + s x_{2\alpha+\beta} + x_{\alpha+\beta} \in G \cdot (x_\beta + x_{2\alpha+\beta})$ for nonzero $s \in k$. Taking closures gives that $x_\beta + x_{\alpha+\beta} \in \overline{G \cdot (x_\beta + x_{2\alpha+\beta})}$. Similarly, using just the torus action, gives that $x_\alpha, x_\beta \in \overline{G \cdot (x_\beta + x_{\alpha+\beta})}$. Also, $\overline{G \cdot x_\alpha} \cap \overline{G \cdot x_\beta} = \{0\}$. The orbit dimensions in $\mathcal{N}_1(G)$ are $(10, 8, 6, 6, 0)$.

(6.5.1) Example. Assume that $p \geqq h$. Given $I \subseteqq \Pi$, let $\lambda \in X(T)_+$ satisfy

$$(\lambda, \alpha^\vee) = \begin{cases} p-1, & \alpha \in I, \\ 0, & \alpha \notin I. \end{cases}$$

Then $\Phi_{\lambda,p} = \Phi_I$ and $V_{G_1}(H^0(\lambda)) = G \cdot \mathfrak{u}_I$. It is well-known that the nilpotent radical $\mathfrak{u}_I$ contains a dense P_I-orbit $\mathcal{O}_I$. If $x \in \mathcal{O}_I$, then any element $y \in G \cdot x$ is called a Richardson element of type P_I. Now suppose that P_J is another parabolic subgroup of G which is associated to P_I, in the sense that the Levi factors L_I and L_J are conjugate in G. This means there exists $y \in W$ such that $y(I) = J$. It follows that $\Phi_{\lambda,p}$ is W-conjugate to Φ_J also. Hence, $G \cdot \mathfrak{u}_J = V_{G_1}(H^0(\lambda)) = G \cdot \mathfrak{u}_I$, so that Richardson elements of type P_I identify with Richardson elements of type P_J. Hence, Richardson elements for any two associated parabolic subgroups are G-conjugate (Johnston-Richardson theorem [JR]). Since our assumption $p \geqq h$ guarantees that p is good, $\mathcal{N}_1(G)$ is G-equivariantly homeomorphic to the unipotent variety of G [SS], (3.12), so the corresponding result for unipotent Richardson elements follows also.

(6.5.2) Example. While not difficult to prove directly, the equality (4.5.2) also has a support variety explanation at least when $p \geqq n = h$. Given $J \subseteqq \Pi$, choose $\lambda \in X(T)_+$ so that $w(\Phi_{\lambda,p}) = \Phi_J$ for some $w \in W$.[8] Then [Jan3], pp. 105–106 proves directly that $\mathrm{GL}_n(k) \cdot x_{\sigma(J)'} \subseteqq V_{(\mathrm{GL}_n)_1}(H^0(\lambda))$. The argument involves showing via a Weyl character argument that, for $I \subseteqq \Pi$ satisfying $\sigma(I) = \sigma(J)'$, $H^0(\lambda)|_{L_I}$ has a direct summand with dimension prime to p. An essential step involves the fact that, for Young subgroups W' and W'' of $W = \mathfrak{S}_n$ corresponding to dual partitions, there exists a unique double coset $W'wW''$ with trivial intersection property, i.e., $W' \cap wW''w^{-1} = \emptyset$—a fact closely related to the isomorphism $S_\mu^* \cong S_{\mu'} \otimes \text{sign}$ for complex Specht modules. On the other hand, $\dim \mathrm{GL}_n(k) \cdot x$ can be easily calculated for any nilpotent matrix x in Jordan normal form (see [SS], p. 251), viz., $\dim \mathrm{GL}_n(k) \cdot x_{\sigma(\lambda)'} = \dim \mathrm{GL}_n(k) \cdot \mathfrak{u}_J$. Now (4.5.2) follows.

6.6. Using (3.5.1) and (5.6.1), we can obtain information on the support varieties when G has type A_2, B_2 or G_2 and p is good. If $I = \{\alpha\} \subseteqq \Pi$, then $G \cdot \mathfrak{u}_I$ has codimension 2 in the variety $\mathcal{N}(G)$ of nilpotent elements in $\mathfrak{g}$. Hence, $G \cdot \mathfrak{u}_I = \bar{\mathcal{O}}_{\text{sub-reg}}$, the closure of the subregular class in $\mathcal{N}(G)$. In type A_2 (resp., B_2; G_2) the dimensions of the G-orbits in $\mathcal{N}(G)$ are $0, 4, 6$ (resp., $0, 4, 6, 8$; $0, 6, 8, 10, 12$). The orbit closures are linearly ordered. (See [Car].)

In the following, let $\{\alpha, \beta\}$ be the simple roots for the root system of G. If there are two root lengths, let α be the short root. Given non-negative integers r, s, let $[r, s]$ denote the dominant weight λ satisfying $(\lambda, \alpha^\vee) = r$ and $(\lambda, \beta^\vee) = s$.

(6.6.1) Corollary. *Let G be a simple algebraic group of rank 2 with p good.*

(a) *Let $\lambda \in X(T)_+$ and assume that $\lambda - (p-1)\rho \notin pX(T)$. If G has type B_2 or G_2, assume that $p \geqq h$. Then*

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[8] Depending on J, the requirement that $p \geqq n$ can be relaxed: the required weight λ will exist precisely when $\sigma(J)$ has at most p nonzero parts.

$$\overline{\mathcal{O}}_{\text{sub-reg}} \subseteqq V_{G_1}(L(\lambda)).$$

If $\Phi_{\lambda,p} \neq \emptyset$ *(i.e.,* λ *is singular), this containment is an equality.*

(b) *If* λ *is regular, then* $V_{G_1}(L(\lambda)) = V_{G_1}$, *except possibly when* G *has type* G_2, $p = 17$ *(resp.,* $p = 29$) *and* λ *lies in the same (restricted) alcove as* $\lambda = [10, 14]$ *(resp.,* $\lambda = [25, 25]$).

(c) *In types* B_2 *(for* $p > 3$) *and* G_2, *there does not exist an irreducible* G*-module* $L(\lambda)$ *such that* $V_{G_1}(L(\lambda))$ *is the closure of the unique non-trivial* G*-orbit of minimal dimension.*

Proof. We can assume that $\lambda \in X_1(T)$. By (5.6.1)(a) and (6.1.1), $V_{G_1}(L(\lambda)) \subseteqq G \cdot \mathfrak{u}_I$, where $w(\Phi_{\lambda,p}) = \Phi_I$ for some $I \subseteqq \Pi$ and some $w \in W$.

First, suppose that $\Phi_{\lambda,p} = \emptyset$. Then $p \nmid \dim L(\mu)$ for some $\mu \in X_1(T)$ lying in the same alcove as λ, except in the exceptional cases for G_2 cited in the statement of (b) [JJ], §§4,5. Thus, assume λ is not one of these exceptional weights. Then using [Jan2], II, §7, (2.2.12) implies that

$$c_{G_1}(L(\lambda)) = c_{G_1T}(L(\lambda)) = c_{G_1T}(L(\mu)) = c_{G_1}(L(\mu)).$$

Since $p \nmid \dim L(\mu)$, we find, using the irreducibility of V_{G_1}, that $V_{G_1}(L(\mu)) = V_{G_1}$. This proves (b).

For arbitrary $\lambda \in X_1(T)$, $p^2 \nmid \dim L(\lambda)$, except when $p = 17$ (resp., $p = 29$) with $\lambda = [10, 14]$ (resp., $[24, 26]$ or $[25, 25]$ which lie in the same alcove). In each of these cases, there is a weight μ lying in the same alcove as these regular weights and which satisfies $p^2 \nmid \dim L(\mu)$; see [JJ], §§4,5. Thus, using a translation argument, we can replace λ with μ to assume $p^2 \nmid \dim L(\lambda)$. Hence, in (3.5.1)(b), taking $M = L(\lambda)$, we obtain

$$|\Phi| - d(\Phi, p) - c_{G_1}(L(\lambda)) \leqq 2.$$

In all cases, $d(\Phi, p) \leqq 2$. Thus, in type G_2, $c_{G_1}(L(\lambda)) \geqq |\Phi| - 4 > 6$. In type B_2 with $p > 3$, $d(\Phi, p) = 0$, so $c_{G_1}(L(\lambda)) \geqq |\Phi| - 2 > 4$. This proves (c). Finally, (a) follows from the same calculation, since $d(\Phi, p) = 0$ for $p \geqq h$. □

In connection with (6.6.1)(b), we refer the reader ahead to Remark (7.4.2) where we indicate how the restriction on $p \neq 17, 29$ can be lifted.

7. Varieties for $Z_r(\lambda)$ and $L(\lambda)$

7.1. A_2-example. We begin with an example which will provide some motivation and insight for the forthcoming results. For $G = \mathrm{GL}_3(k)$, we compare the support varieties of $Z_1(\lambda)$ and $H^0(\lambda)$. Write $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. For each positive root α, let $y_\alpha = x_{-\alpha}$ be the negative root vector associated to α. Recall that $\overline{G \cdot y_{\alpha_1}} = \overline{G \cdot y_{\alpha_2}} = \overline{G \cdot y_{\alpha_1+\alpha_2}} = \overline{\mathcal{O}}_{\text{sub-reg}}$.

(7.1.1) Proposition. *Let* $G = \mathrm{GL}_3(k)$, $\lambda \in X_1(T)$ *with* $p \geqq h = 3$.

(a) *If* λ *is* p*-regular then* $V_{G_1}(H^0(\lambda)) = \mathcal{N}_1(G)$ *and* $V_{G_1}(Z_1(\lambda)) = \mathcal{N}_1(U)$.

(b) *If* $(\lambda+\rho,\alpha^\vee)\in p\mathbb{Z}$ *for* $\alpha=\alpha_1$, *but not for* $\alpha=\alpha_2,\alpha_1+\alpha_2$, *then* $V_{G_1}\big(H^0(\lambda)\big)=\overline{G\cdot y_{\alpha_1}}$ *and* $V_{G_1}\big(Z_1(\lambda)\big)=\overline{B\cdot y_{\alpha_2}}$.

(c) *If* $(\lambda+\rho,\alpha^\vee)\in p\mathbb{Z}$ *for* $\alpha=\alpha_2$, *but not for* $\alpha=\alpha_1,\alpha_1+\alpha_2$, *then* $V_{G_1}\big(H^0(\lambda)\big)=\overline{G\cdot y_{\alpha_1}}$ *and* $V_{G_1}\big(Z_1(\lambda)\big)=\overline{B\cdot y_{\alpha_1}}$.

(d) *If* $(\lambda+\rho,\alpha^\vee)\in p\mathbb{Z}$ *for* $\alpha=\alpha_1+\alpha_2$, *but not for* $\alpha=\alpha_1,\alpha_2$, *then* $V_{G_1}\big(H^0(\lambda)\big)=\overline{G\cdot y_{\alpha_1}}$ *and* $V_{G_1}\big(Z_1(\lambda)\big)=\overline{B\cdot y_{\alpha_1}}\cup\overline{B\cdot y_{\alpha_2}}$.

(e) *If* $(\lambda+\rho,\alpha^\vee)\in p\mathbb{Z}$ *for* $\alpha\in\Phi^+$, *then* $V_{G_1}\big(H^0(\lambda)\big)=\{0\}=V_{G_1}\big(Z_1(\lambda)\big)$.

Proof. The computation of the $V_{G_1}\big(H^0(\lambda)\big)$ follows from (6.6.1). Next, (a) follows from by (4.1.1); (e) holds because, under the hypotheses, $Z_1(\lambda)$ is a projective G_1-module.

The proper B-orbits in $\mathcal{N}_1(U)$ are given by $\{0\}$, $\overline{B\cdot y_{\alpha_1}}$, $\overline{B\cdot y_{\alpha_2}}$, and $\overline{B\cdot y_{\alpha_1+\alpha_2}}$ with $\overline{B\cdot y_{\alpha_1+\alpha_2}}\subset\overline{B\cdot y_{\alpha_j}}$ for $j=1,2$. Suppose that $(\lambda+\rho,\alpha^\vee)\in p\mathbb{Z}$ for $\alpha=\alpha_1$, but not for $\alpha=\alpha_2,\alpha_1+\alpha_2$. By (4.3.1), $y_{\alpha_1}\notin V_{G_1}\big(Z_1(\lambda)\big)$ and $y_{\alpha_2}\in V_{G_1}\big(Z_1(\lambda)\big)$. Also, by (4.1.1), $V_{G_1}\big(Z_1(\lambda)\big)\neq\mathcal{N}_1(U)$. Hence, $V_{G_1}\big(Z_1(\lambda)\big)=\overline{B\cdot y_{\alpha_2}}$, so (b) holds. Now (c) follows in a similar manner. The hypotheses of (d) imply that $\overline{B\cdot y_{\alpha_1}}\cup\overline{B\cdot y_{\alpha_2}}\subseteq V_{G_1}\big(Z_1(\lambda)\big)$. Once again by (4.1.1), $V_{G_1}\big(Z_1(\lambda)\big)\neq\mathcal{N}_1(U)$, thus $\overline{B\cdot y_{\alpha_1}}\cup\overline{B\cdot y_{\alpha_2}}=V_{G_1}\big(Z_1(\lambda)\big)$. ☐

7.2. The results from the preceding section indicate that number of irreducible components of $V_{G_r}\big(Z_r(\lambda)\big)$ can vary depending on the position of λ. Let σ be the involutory graph automorphism of G such that $\sigma|_T$ induces the opposition involution $-w_0$ on $X(T)$ where $w_0\in W$ is the long word. We can assume that σ is defined over $\mathbb{F}_p$, so it induces an automorphism on G_r, B_r, V_{G_r}, etc. The following result indicates how σ permutes the support varieties of the $Z_r(\lambda)$.

(7.2.1) Proposition. *If* $\lambda\in X(T)$ *then* $V_{G_r}\big(Z_r(\lambda)\big)=\sigma\cdot V_{G_r}\big(Z_r(w_0\cdot\lambda)\big)$.

Proof. For a G_r-module M, let M^σ be the G_r-module with action twisted by σ. Then $Z_r(\lambda)^\sigma\cong Z_r(-w_0\lambda)$. We have $Z_r(\lambda)^*\cong\operatorname{ind}_{B_r}^{G_r}(-\lambda+2(p^r-1)\rho)$; cf. [Jan2], II, (3.5). Consequently,

$$\begin{aligned}\big(Z_r(\lambda)^\sigma\big)^*&\cong Z_r(-w_0\lambda)^*\\&\cong Z_r\big(w_0\lambda+2(p^r-1)\rho\big)\\&\cong Z_r(w_0\cdot\lambda)\otimes 2p^r\rho,\end{aligned}$$

since $w_0(\rho)=-\rho$. Since $V_{G_r}(M)=V_{G_r}(M^*)$ for any finite dimensional G_r-module, we have (using [NP], (2.1) for the last equality):

$$V_{G_r}\big(Z_r(w_0\cdot\lambda)\big)=V_{G_r}\big(Z_r(\lambda)^\sigma\big)=\sigma^{-1}\cdot V_{G_r}\big(Z_r(\lambda)\big).$$

This completes the proof. ☐

7.3. We next consider the following result when $r=1$.

(7.3.1) Theorem. *Let* G *be a reductive algebraic group and let* $\lambda\in X(T)_+$. *Assume that* p *is good. Then*

$$V_{G_1}\big(H^0(\lambda)\big) = G\cdot V_{G_1}\big(Z_1(w\cdot\lambda)\big)$$

for all $w \in W$. *Equivalently*

$$G\cdot V_{G_1}\big(Z_1(\lambda)\big) = G\cdot V_{G_1}\big(Z_1(w\cdot\lambda)\big)$$

for all $w \in W$.

Proof. According to (6.2.1), we have

$$V_{G_1}\big(H^0(\lambda)\big) = V_{G_1}\big(H^0(\mu)\big) \quad \text{for all } \mu \in \Gamma_1(\lambda) \cap X(T)_+.$$

Therefore, by (4.6.1),

$$V_{G_1}\big(H^0(\lambda)\big) = G\cdot\left[\bigcup_{\mu\in\Gamma_1(\mu)} V_{G_1}\big(Z_r(\mu)\big)\right] = G\cdot\left[\bigcup_{w\in W} V_{G_1}\big(Z_1(w\cdot\lambda)\big)\right].$$

It follows that $G\cdot V_{G_1}\big(Z_1(w\cdot\lambda)\big) \subseteq V_{G_1}\big(H^0(\lambda)\big)$ for all $w \in W$. The other inclusion follows by (5.6.1). □

One should observe that even though $G\cdot V_{G_1}\big(Z_1(\mu_1)\big) = G\cdot V_{G_1}\big(Z_1(\mu_2)\big)$ for all $\mu_1, \mu_2 \in \Gamma_1(\lambda)$, one may have $V_{G_1}\big(Z_1(\mu_1)\big) \neq V_{G_1}\big(Z_1(\mu_2)\big)$ for some $\mu_1, \mu_2 \in \Gamma_1(\lambda)$ from the example given in (7.1.1).

7.4. In connection with (5.6.1) again, the proposition below follows immediately from (6.2.1), (6.2.2). Another proof can be easily given using induction on λ, (3.4.2), (6.2.1) and the linkage principle.

(7.4.1) Proposition. *Let* G *be a reductive algebraic group such that* p *is good. Let* $\lambda \in X(T)_+$ *and choose* $w \in W$ *such that* $w(\Phi_{\lambda,p}) = \Phi_I$ *for some* $I \subseteq \Pi$. *Then*

$$V_{G_1}\big(L(\lambda)\big) \subseteq G\cdot\mathfrak{u}_I.$$

(7.4.2) Remark. Assume that $\lambda \in X(T)_+$ is regular and write $\lambda = \lambda_0 + p\lambda_1$, where $\lambda_0 \in X_1(T)$ is a regular p-restricted. By the tensor product theorem, $L(\lambda) \cong L(\lambda_0)\otimes L(\lambda_1)^{(1)}$, so $V_{G_1}\big(L(\lambda)\big) = V_{G_1}\big(L(\lambda_0)\big)$. Thus, to calculate the support variety of $L(\lambda)$, we can assume that λ is restricted itself, and, by a simple translation argument, that $\lambda = w\cdot 0$ for some $w \in W_p$. Now assume that $p > h$ and the Lusztig conjecture holds for the characters of irreducible G-modules $L(\mu)$, for dominant weights μ in the Janzten region Γ_J, i.e., $(\mu+\rho,\alpha^\vee) \leq p(p-h+2)$. It is known that $X_1(T) \subseteq \Gamma_J$ provided $p \geq 2h-3$. Assume that our $\lambda \in \Gamma_J$. We sketch a proof below that $V_{G_1}\big(L(\lambda)\big) = V_{G_1}$. This result is attributed to J. C. Janzten by the referee. We have slightly modified the suggested argument.

It suffices to show the stronger result that the relative support variety $V_{G_1}\big(L(w\cdot 0), k\big)$ equals V_{G_1}. In the bounded derived category $D^b(G\text{-mod})$ of rational G-modules, the complex $L(w\cdot 0)[-l(w)]$ (i.e., the complex which is identically 0 except for $L(w\cdot 0)$ concentrated in degree $l(w)$) has a filtration (in a sense made precise in [CPS3], p. 515) with sections of the form $H^0(x \cdot 0)[2m - l(x)]$, $xw_0 \leqq ww_0$, $m \in \mathbb{Z}$, where w_0 is the long word in W. In fact, this statement is equivalent (by [CPS3], (5.3)) to the truth of the Lusztig conjecture would imply that

$$[L(w \cdot 0)] = \sum_{xw_0 \leqq ww_0} (-1)^{l(w)-l(x)} p_{x,w}[H^0(x \cdot 0)],$$

in the Grothendieck group of G-mod, where the $p_{x,w}$ are nonnegative integers and w_0 is the long word in W. Necessarily, $p_{x,w}$ is the number of terms of the form $H^0(x \cdot 0)[2m - l(x)]$ in the filtration of $L(w \cdot 0)[-l(w)]$.

Let $R = H^\bullet(G_1, k) = H^{2\bullet}(G_1, k)$, and let $K(R)$ be the Grothendieck group of R-modules. For a G-module M, let $\mathscr{M}^+ = \bigoplus_{i \in 2\mathbb{N}} H^i(G_1, M)$ and $\mathscr{M}^- = \bigoplus_{i \in 2\mathbb{N}} H^{i+1}(G_1, M)$. Put $\chi(M) = [\mathscr{M}^+] - [\mathscr{M}^-]$, where $[\mathscr{M}^\pm]$ denotes the image of $\mathscr{M}^\pm$ in $K(R)$.

By [KLT], Thm. 2, Thm. 8 (which improves upon [AJ], (3.6), (3.8)),

$$(-1)^{l(x)}\chi(H^0(x \cdot 0))$$

is a class of an actual (not virtual) object in R-mod whose support consists of all of V_{G_1} since $x \cdot 0$ is regular. In other words, $H^i(G_1, H^0(x \cdot 0))$ vanishes unless $i \equiv l(x) \mod 2$. Taking G_1-cohomology gives that

$$(-1)^{l(w)}\chi(L(w \cdot 0)) = \sum p_{x,w}(-1)^{l(x)}\chi(H^0(x \cdot 0)).$$

So $(-1)^{l(w)}\chi(L(w \cdot 0))$ is a sum with positive coefficients of classes of R-modules in which the support of each summand is V_{G_1}. Finally, for any maximal ideal $\mathfrak{m}$ in R, localization at $\mathfrak{m}$ is an exact functor from R-mod to $R_\mathfrak{m}$-mod, so it defines a natural homomorphism $(-)_\mathfrak{m}: K(R) \to K(R_\mathfrak{m})$. Thus, for any $\mathfrak{m}$, $H^\bullet(G_1, L(w \cdot \lambda))_\mathfrak{m} \neq 0$, as required.

In particular, since the Lusztig conjecture holds for all rank 2 groups, the restriction, in (6.6.1)(b), that $p \neq 17$ or 29 can be dropped. In fact, the above argument proves (6.6.1)(b) as long as $p > 7$. In the spirit of (7.3.2), it would be interesting to compare $V_{G_1}(L(\lambda))$ and $V_{G_1}(H^0(\lambda))$ for general dominant weights λ. We leave this as an open question.

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Department of Mathematics, University of Georgia, Athens, GA 30602, U.S.A.
e-mail: nakano@math.uga.edu

Department of Mathematics, University of Virginia, Charlottesville, VA 22903-3199, U.S.A.
e-mail: bjp8w@fermi.clas.virginia.edu

Department of Mathematics and Computer Science, Skidmore College, Saratoga Springs, NY 12866, U.S.A.
e-mail: dvella@skidmore.edu

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